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INDEX THEOREMS FOR
THE PROBLEM OF BOLZA IN THE
CALCULUS OF VARIATIONS

A DISSERTATION SUBMITTED TO THE
FACULTY OF THE DIVISION OF THE
PHYSICAL SCIENCES IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1940

By

KATHARINE ELIZABETH HAZARD

Private Edition, Distributed by
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INDEX THEOREMS FOR THE PROBLEM OF BOLZA
IN THE CALCULUS OF VARIATIONS

1. Introduction. In the present paper we shall study index theorems of the problem in the calculus of variations determined by a functional of the form

$$(1:1) \quad J(\eta) = 2q(\eta) + \int_{x_1}^{x_2} 2\omega(\eta, \eta') dx$$

in $(x, \eta_1, \dots, \eta_n)$ -space subject to a set of end conditions

$$(1:2) \quad \Psi_{\mu}(\eta) = a_{\mu 1} \eta_1(x_1) + b_{\mu 1} \eta_1(x_2) = 0$$

($\mu = 1, \dots, p < 2n$)

and to a set of differential equations

$$(1:3) \quad \Phi_{\nu}(\eta, \eta') = S_{\nu 1}(x) \eta_1 + T_{\nu 1}(x) \eta'_1 = 0$$

($\nu = 1, \dots, m < n$).

In the functional (1:1) the integrand 2ω has the form

$$(1:4) \quad 2\omega = P_{1k}(x) \eta_1 \eta_k + 2Q_{1k}(x) \eta_1 \eta'_k + R_{1k}(x) \eta'_1 \eta'_k$$

($1, k = 1, \dots, n$)

and $2q$ is a symmetric quadratic form in the end values $\eta_1(x_1)$, $\eta_1(x_2)$. A problem of this type is suggested by the study of the second variation for the general problems of Bolza and Lagrange and is usually called the accessory problem. In the present paper we shall generally denote this problem by the symbol P .

A large portion of the theory developed in the following pages is an extension to the problem of Bolza of the theory developed by Birkhoff and Hestenes (I, 200-243)* for the case in which the differential equations (1:3) are absent. Some of the results here given, particularly those in Section 3, were obtained also by Hestenes (VI) in an unpublished paper. Frequent use has been made of the types of proofs used in these papers. However, many modifications and simplifications have been introduced and in many cases the approach is entirely different from that of Birkhoff and Hestenes. Most of the simplifications here made are a consequence of an effective use of the relationships existing between problems and sub-problems. A connection between the index theory here presented and the index forms used by Morse is found in Section 7 below. An excellent discussion of boundary value problems and their relationships to index theorems in the calculus of variations has been given recently by Reid (XI).

In Section 2 certain fundamental definitions and assumptions essential to the analysis of later sections are given. In Section 3 the index of the problem P is defined as the dimensionality of a maximal linear subspace of the space of arcs under consideration on which $J(\gamma)$ is negative definite. Various properties of the index are obtained, including its relationships to the natural isoperimetric conditions and the minimal sets used by Birkhoff and Hestenes.

In Section 4 the general notion of a sub-problem is introduced. By the use of such problems much of the theory of later sections is developed. In particular the well-known

* Roman numerals in parenthesis refer to the bibliography at the end of the paper. The Arabic numerals refer to pages.

oscillation and comparison theorems given in Section 8 are proved by means of sub-problems. The concept of index of a problem with respect to a sub-problem is defined in Section 5.

In Section 6 the one variable end point problem is discussed and in view of certain abnormality conditions which may arise it has been convenient to introduce the notion of focal intervals. These intervals play much the same role that focal points play in the simple problem. In particular it is shown that the index is equal to the number of focal intervals interior to the interval x_1x_2 .

A special type of sub-problem is discussed in Section 7. This problem is obtained, in brief, by requiring the arcs associated with it to vanish at a certain prescribed set of points on x_1x_2 . By properly choosing this set of points and making use of the properties of the index of a problem with respect to a sub-problem it is shown that the definition of index given in Section 3 is equivalent to the definition given by Morse in terms of broken extremal arcs.

In Section 9 there is a very brief discussion of the boundary value problem including the well-known result that the index is equal to the number of negative characteristic values of the associated boundary value problem. In Section 10 a pseudo-periodic problem is considered and several interesting notions including the frequency number are discussed. Also the equivalence of the index of repletion as defined by Morse and Pitcher (X) and the index of a problem with respect to a sub-problem is shown.

2. Preliminary definitions and assumptions. We shall assume in the quadratic form (1:4) that the functions $P_{1k} = P_{k1}$ are continuous and the functions Q_{1k} and $R_{1k} = R_{k1}$ are of class C' on the interval x_1, x_2 . By a function being of class C^n we mean that it is continuous and possesses continuous derivatives of the first n orders. Also we assume that the matrix $\|a_{\mu 1}, b_{\mu 1}\|$ of the coefficients in (1:2) has rank p , that is, that the conditions (1:2) are linearly independent.

An arc $\gamma_1(x)$ will be said to be of class D' if it is continuous and consists of a finite number of sub-arcs each of which is of class C' . Arcs of class D' will be called admissible arcs. Let $\mathcal{A}$ denote the set of admissible arcs and let $\mathcal{D}$ denote the set of all arcs γ in $\mathcal{A}$ satisfying the differential equations (1:3). The functions $S_{\nu 1}$ in these equations are assumed to be continuous and the functions $T_{\nu 1}$ of class C' . Let $\mathcal{E}$ denote the sub class of $\mathcal{D}$ consisting of all arcs γ in $\mathcal{D}$ satisfying the end conditions (1:2). It is to be noted that the set $\mathcal{E}$ is a linear set in the sense that if γ and ξ are any two arcs in $\mathcal{E}$ and a, b are arbitrary real numbers, then the arc $a\gamma + b\xi$ is also in $\mathcal{E}$. The sets $\mathcal{A}$ and $\mathcal{D}$ also form linear sets in this sense.

The Euler equations and transversality conditions for the accessory problem determined by (1:1) and the set of arcs are

$$(2:1) \quad L_1(\gamma, \gamma', \lambda) = \Omega_{\gamma_1} - \frac{d}{dx} \Omega_{\gamma'_1} = 0,$$

$$(2:2) \quad T_{11}(\gamma, \lambda, \lambda) = q_{\gamma_{11}} + \lambda_{\mu} a_{\mu 1} - \Omega_{\gamma'_1}(x_1) = 0,$$

$$T_{12}(\gamma, \lambda, \lambda) = q_{\gamma_{12}} + \lambda_{\mu} b_{\mu 1} + \Omega_{\gamma'_1}(x_2) = 0,$$

where $\Omega(\gamma, \gamma') = \omega(\gamma, \gamma') + \lambda_{\nu}(x) \Phi_{\nu}(\gamma, \gamma')$ and the symbol

γ_1 , for example, stands for $\gamma_1(x_1)$. A solution $\gamma_1(x)$, $\lambda_\nu(x)$ of equations (1:3) and (2:1) where the functions $\gamma_1(x)$ are of class C^n and the multipliers $\lambda_\nu(x)$ are of class C^1 is called an accessory extremal or, simply, an extremal. If the determinant

$$R = \begin{vmatrix} R_{1k} & T_{\nu 1} \\ T_{\tau k} & 0 \end{vmatrix} \quad (1, k = 1, \dots, n; \tau, \nu = 1, \dots, m)$$

is different from zero then an extremal may be expressed by the canonical variables γ_1, ζ_1 where $\zeta_1 = \Omega \gamma'_1$. The canonical equations equivalent to (2:1) are

$$\gamma'_1 = A_{1k} \gamma_k + B_{1k} \zeta_k, \quad \zeta'_1 = C_{1k} \gamma_k - A_{k1} \zeta_k,$$

where the coefficients A_{1k}, B_{1k}, C_{1k} are continuous functions of x , (VII, 13-14). We shall assume throughout the following pages that R is different from zero and hence may use the canonical variables for an extremal whenever it seems more convenient to do so. An extremal which satisfies the transversality conditions (2:2) with a set of constants λ_μ will be called a transversal extremal. It will often be convenient to consider only the functions $\gamma_1(x)$ associated with an extremal γ_1, ζ_1 . Such a set of functions γ_1 will be said to form an extremal arc. It is important to observe that the terms "extremal arc" and "extremal" here defined are used to denote different concepts, the extremal arc being the arc $\gamma_1(x)$ ($x_1 \leq x \leq x_2$) associated with the extremal $\gamma_1(x), \zeta_1(x)$ ($x_1 \leq x \leq x_2$). It is also to be noted that an extremal arc may be associated with more than one extremal γ_1, ζ_1 . The functions $\gamma_1(x)$ associated with a transversal extremal will be called a transversal arc.

The Clebsch condition will be said to hold on $x_1 x_2$ if the inequality

$$(2:3) \quad R_{1k}(x) \pi_1 \pi_k \geq 0$$

is true for every set of constants $(\pi) \neq (0)$ satisfying the equations

$$T_{\nu 1}(x) \pi_1 = 0,$$

where R_{1k} and $T_{\nu 1}$ are defined as in equations (1:4) and (1:3) respectively. We shall say that the strengthened Clebsch condition holds if the condition (2:3) holds without the equality sign.

It may be noted that the accessory problem P can be formulated in different ways. For example the functional $J(\eta)$ may be of the form

$$J(\eta, u) = b_{h\lambda} u_h u_\lambda + \int_{x_1}^{x_2} 2\omega(\eta, \eta') dx \quad (h, \lambda = 1, \dots, r)$$

and subject to the conditions (1:3) and end conditions

$$\zeta_1(x_s) - c_{1h}^s u_h = 0 \quad (i = 1, \dots, n; h = 1, \dots, r; s = 1, 2),$$

where the matrix $||b_{h\lambda}||$ is symmetric and u_h , c_{1h}^s are constants. That this problem is equivalent to a problem of the type defined in Section 1 in a space of $n + r + 1$ dimensions is seen by adding the differential equations $u_h' = \zeta_{n+h}' = 0$.

3. Index of $J(\eta)$ on $\mathcal{C}$ and minimal sets. Let $\xi_1(x)$ be an arc in $\mathcal{C}$ and define $J(\xi, \eta)$ to be the bilinear functional

$$(3:1) \quad \begin{aligned} J(\xi, \eta) = & \xi_{11}^0 \eta_{11} + \xi_{12}^0 \eta_{12} \\ & + \int_{x_1}^{x_2} (\xi_1 \omega_{\eta_1} + \xi_1' \omega_{\eta_1'}) dx. \end{aligned}$$

The condition

$$J(\xi, \eta) = 0$$

will be called a natural isoperimetric condition for the functional $J(\gamma)$ on $\mathcal{E}$.

We now prove the following

THEOREM 3:1. An arc γ in $\mathcal{D}$, that is, an admissible arc satisfying the differential equations (1:3), is a transversal arc if and only if γ satisfies the condition $J(\xi, \gamma) = 0$ for every arc ξ in $\mathcal{E}$.

For suppose γ is a transversal arc and let λ_ν, λ_μ be a set of multipliers associated with γ such that the equations (2:1) and (2:2) are satisfied. Since $J(\xi, \gamma)$ is expressible in the form

$$J(\xi, \gamma) = \xi_{11} q_{\gamma 11} + \xi_{12} q_{\gamma 12} + \int_{x_1}^{x_2} [\xi_1 \Omega_{\gamma 1} + \xi'_1 \Omega_{\gamma'_1} + \lambda_\nu \Phi_\nu(\xi, \xi')] dx$$

we have, after adding $\lambda_\mu \Psi_\mu(\xi)$ to both sides and integrating by parts, that

$$\begin{aligned} J(\xi, \gamma) + \lambda_\mu \Psi_\mu(\xi) &= \xi_{11} [q_{\gamma 11} + \lambda_\mu a_{\mu 1} - \xi_1(x_1)] \\ &+ \xi_{12} [q_{\gamma 12} + \lambda_\mu b_{\mu 1} + \xi_1(x_2)] \\ &+ \int_{x_1}^{x_2} [\xi_1 L_1(\gamma) + \lambda_\nu \Phi_\nu(\xi, \xi')] dx, \end{aligned}$$

where $\xi_1 = \Omega_{\gamma'_1}$ and $L_1(\gamma)$ are the functions defined in (2:1).

If ξ is any arc in $\mathcal{E}$ then $\Psi_\mu(\xi) = 0$ and the constants λ_μ may be chosen so that the quantities in the first two brackets are equal respectively to the functions T_{11}, T_{12} in (2:2). With the help of equations (2:1) it follows that $J(\xi, \gamma) = 0$.

Conversely, suppose $J(\xi, \gamma) = 0$ for all arcs ξ in $\mathcal{E}$.

Let ξ_α ($\alpha = 1, \dots, r$) be a set of arcs in $\mathcal{D}$ such that the matrix $\|\Psi_\mu(\xi_\alpha)\|$ has maximum rank r . It follows that the matrix

$$\left\| \begin{array}{cc} J(\xi_\alpha, \eta), & J(\xi, \eta) \\ \Psi_\mu(\xi_\alpha), & \Psi_\mu(\xi) \end{array} \right\|$$

also has rank r for all arcs ξ in $\mathcal{L}$. To show this we note that there exist constants a_α such that $\Psi_\mu(\xi + a_\alpha \xi_\alpha) = 0$. Hence the arc $\xi + a_\alpha \xi_\alpha$ is in $\mathcal{E}$ and $J(\xi + a_\alpha \xi_\alpha, \eta) = 0$. The above matrix accordingly has rank r for every arc ξ in $\mathcal{L}$. It now follows that there exist constants b_μ such that the equation

$$J(\xi, \eta) + b_\mu \Psi_\mu(\xi) = 0$$

holds for all arcs ξ in $\mathcal{L}$. The remaining part of the argument is similar to that given by Bliss in the proof of the Multiplier Rule for the problem of Lagrange and will not be given here (III, 23-26).

We now establish the result

THEOREM 3:2. The number of linearly independent arcs in a maximal set of arcs in $\mathcal{E}$, every proper linear combination η of which gives $J(\eta)$ a negative value, is the same for every such set.

In order to prove this let ξ_α ($\alpha = 1, \dots, k$), η_r ($r = 1, \dots, k'$) be two such maximal sets of arcs in $\mathcal{E}$. If $k < k'$ there exist constants c_r , not all zero, such that $J(\xi_\alpha, c_r \eta_r) = 0$. It then follows that every linear combination η of the arcs $\xi_\alpha, c_r \eta_r$ gives $J(\eta)$ a negative value which contradicts the assumption on the ξ_α . By symmetry k cannot be greater than k' . Hence $k = k'$.

If the number of linearly independent arcs in a maximal set of arcs in $\mathcal{E}$ every proper linear combination η of which gives $J(\eta)$ a negative value is finite, then this number is

defined to be the index of $J(\gamma)$ on $\mathcal{E}$. If the number of such arcs is not finite the index is said to be infinite. In Section 7 it will be shown that if the strengthened Clebsch condition holds then the index is always finite. We shall assume that the index is finite in this and following sections except, of course, in the proof of the above fact.

A set of k natural isoperimetric conditions

$$(3:2) \qquad J(\xi_\alpha, \gamma) = 0 \qquad (\alpha = 1, \dots, k)$$

will be called a proper set if the determinant $|J(\xi_\alpha, \xi_\beta)|$ is different from zero. The set (3:2) will be called a minimal set if the inequality $J(\gamma) \geq 0$ is true for every arc γ in $\mathcal{E}$ satisfying these conditions and if there exists no set of the form (3:2) with fewer conditions having this property. This differs from the definition of a minimal set as stated by Birkhoff and Hestenes, however, an examination of the proofs of their theorems shows that the definition used is that stated here. If the set (3:2) is a minimal set then the set $J(\bar{\xi}_\alpha, \gamma) = 0$ is also a minimal set where $\bar{\xi}_\alpha = \xi_\beta A_{\alpha\beta}$, the A 's being constants with $|A_{\alpha\beta}| \neq 0$. As a consequence of this fact we may suppose that the arcs ξ_α are chosen so that

$$(3:3) \qquad J(\xi_\alpha, \xi_\beta) = 0 \qquad (\alpha \neq \beta; \alpha, \beta = 1, \dots, k).$$

Concerning minimal sets we have the following useful theorems.

THEOREM 3:3. Let $J(\xi_\alpha, \gamma) = 0$ ($\alpha = 1, \dots, k$) be a proper minimal set defined by arcs ξ_α in $\mathcal{E}$. If there exists an arc ξ in $\mathcal{E}$ such that $J(\xi) = J(\xi_\alpha, \xi) = 0$, then ξ is a transversal arc.

The theorem follows from Theorem 3:1 when it is shown that $J(\xi, \eta) = 0$ for every arc η in $\mathcal{E}$. Let η be any arc in $\mathcal{E}$. If $J(\xi_\alpha, \eta) = 0$ then the equations

$$J(\xi_\alpha, \eta + b\xi) = J(\xi_\alpha, \eta) + bJ(\xi_\alpha, \xi) = 0$$

hold for all b . Consequently the inequality

$$0 \leq J(\eta + b\xi) = J(\eta) + bJ(\eta, \xi)$$

is true for all b and hence $J(\eta, \xi) = 0$. If $J(\xi_\alpha, \eta) \neq 0$ then, since $|J(\xi_\alpha, \xi_\beta)| \neq 0$, we can select constants a_α ($\alpha = 1, \dots, k$) such that the arc $\bar{\xi} = \eta - \xi_\alpha a_\alpha$ satisfies $J(\xi_\alpha, \bar{\xi}) = 0$. By the preceding argument

$$0 = J(\bar{\xi}, \xi) = J(\eta, \xi) - a_\alpha J(\xi_\alpha, \xi) = J(\eta, \xi).$$

Hence the theorem follows.

THEOREM 3:4. If the set (3:2) is a minimal set defined by arcs ξ_α ($\alpha = 1, \dots, k$) in $\mathcal{E}$, then no arc of the form $\bar{\xi} = \xi_\alpha a_\alpha$, where the a 's are constants not all zero, is a transversal arc. Moreover, the inequality $J(\bar{\xi}) \leq 0$ is true for every arc of this form, and if the set (3:2) is a proper minimal set the strict inequality $J(\bar{\xi}) < 0$ holds.

To prove the first statement of the theorem suppose there exists an arc in $\mathcal{E}$ of the form $\bar{\xi} = \xi_\alpha a_\alpha$ which is a transversal arc. At least one of the a_α , say a_k , is different from zero. Now let η be any arc in $\mathcal{E}$ satisfying the conditions $J(\xi_\beta, \eta) = 0$ ($\beta = 1, \dots, k-1$). Since

$$0 = J(\bar{\xi}, \eta) = J(\xi_\alpha a_\alpha, \eta) = J(\xi_k a_k, \eta) \quad (\alpha = 1, \dots, k)$$

it follows that $J(\eta) \geq 0$. The set (3:2), therefore cannot be

minimal for the set $J(\xi_\alpha, \gamma) = 0$ contains fewer than k conditions.

Now suppose that the arcs ξ_α have been chosen so that the conditions (3:3) hold and that one of the numbers $J(\xi_\alpha, \xi_k)$, say $J(\xi_k, \xi_k)$, is positive. Let γ be an arc in $\mathcal{E}$ satisfying only the first $(k-1)$ conditions (3:2) and having $J(\gamma) < 0$. There exists a constant b such that

$$J(\xi_k, \gamma) = bJ(\xi_k, \xi_k) = bJ(\xi_k).$$

The arc $\gamma - b\xi_k$ then satisfies all the conditions (3:2). But we have

$$\begin{aligned} J(\gamma - b\xi_k) &= J(\gamma) - 2bJ(\xi_k, \gamma) + b^2J(\xi_k) \\ &= J(\gamma) - b^2J(\xi_k) < 0 \end{aligned}$$

which is contrary to our assumption that the set (3:2) is a minimal set. If $|J(\xi_\alpha, \xi_\beta)| \neq 0$ and the conditions (3:3) hold, then clearly every arc of the form $\xi = a_\alpha \xi_\alpha$ has $J(\xi) < 0$.

As a converse to the first part of this theorem we have

THEOREM 3:5. Let ξ_α ($\alpha = 1, \dots, k$) be a maximal set of arcs in $\mathcal{E}$ having the following properties: (1) every arc of the form $\xi = \xi_\alpha a_\alpha$, where the a 's are constants not all zero, satisfies the inequality $J(\xi) \leq 0$, (2) no arc of the form $\xi = \xi_\alpha a_\alpha$ ($a \neq 0$) is a transversal arc. Then the set $J(\xi_\alpha, \gamma) = 0$ forms a minimal set.

For suppose there exists an arc γ in $\mathcal{E}$ satisfying the conditions $J(\xi_\alpha, \gamma) = 0$ and having $J(\gamma) < 0$. Then every linear combination ξ of the $(k+1)$ linearly independent arcs ξ_α, γ gives $J(\xi)$ a negative value and no proper linear combination of these arcs is a transversal arc since $J(\gamma) < 0$. This contradicts the assumption on the ξ_α . Hence every arc γ in $\mathcal{E}$ satisfying $J(\xi_\alpha, \gamma) = 0$ has $J(\gamma) \geq 0$. Now suppose there exist linearly

independent arcs γ_r ($r = 1, \dots, k'$) such that the conditions $J(\gamma_r, \gamma) = 0$ ($r = 1, \dots, k' < k$) form a proper minimal set. Since $k' < k$ there exist solutions (b) $\neq (0)$ of the equations $J(\gamma_r, f_\alpha) b_\alpha = 0$. Hence the arc $f = f_\alpha b_\alpha$ has $J(f) \geq 0$, but by assumption $J(f) \leq 0$. Hence $J(f) = 0$. As a consequence of Theorem 3:3 the arc f is a transversal arc contrary to our assumption. The statement of the theorem now follows.

As a converse to the last statement of Theorem 3:4 we have

THEOREM 3:6. If the variations f_α ($\alpha = 1, \dots, k$) form a maximal set of linearly independent arcs in $\mathcal{E}$, every proper linear combination γ of which gives $J(\gamma)$ a negative value, then the set $J(f_\alpha, \gamma) = 0$ is a proper minimal set.

By the type of argument used in the proof of Theorem 3:5 it follows that every arc γ in $\mathcal{E}$ satisfying the conditions $J(f_\alpha, \gamma) = 0$ has $J(\gamma) \geq 0$. Moreover there can exist no minimal set with fewer than k conditions for then there would exist an arc of the form $f = f_\alpha a_\alpha$ having $J(f) \geq 0$. The set $J(f_\alpha, \gamma) = 0$ is proper for if it were not the determinant $|J(f_\alpha, f_\beta)|$ would be equal to zero and there would exist solutions (b) $\neq (0)$ of the equations $J(f_\alpha, f_\beta) b_\alpha = 0$ which is impossible.

The following theorem gives a useful relationship between minimal sets and the index of $J(\gamma)$ on $\mathcal{E}$.

THEOREM 3:7. If the set $J(f_\alpha, \gamma) = 0$ ($\alpha = 1, \dots, k$) is a minimal set, then k is the index of $J(\gamma)$ on $\mathcal{E}$.

For, assume that the index of $J(\gamma)$ on $\mathcal{E}$ is k' and let γ_r ($r = 1, \dots, k'$) be a maximal set of linearly independent arcs in $\mathcal{E}$ such that every arc of the form $\gamma = \gamma_r a_r$, where the a 's are constants not all zero, satisfies the inequality $J(\gamma) < 0$. Then every solution f of $J(\gamma_r, f) = 0$ has $J(f) \geq 0$. As a consequence of the definition of minimal sets k cannot be greater

than k' . If $k' > k$ then there exist solutions $(a) \neq (0)$ of the equations $J(\xi_\alpha, \gamma) a_\gamma = 0$ ($\alpha = 1, \dots, k$). The set $J(\xi_\alpha, \gamma) = 0$ being a minimal set implies that $J(\gamma, a_\gamma) \geq 0$ contrary to the choice of the γ . Hence $k = k'$ and the conclusion of the theorem is immediate.

COROLLARY 1. The number of conditions in a minimal set is always the same.

COROLLARY 2. If the conditions $J(\xi_\alpha, \gamma) = 0$ ($\alpha = 1, \dots, k$) form a minimal set, then there exists a proper minimal set containing the same number of conditions.

COROLLARY 3. The index of $J(\gamma)$ on $\mathcal{E}$ is equal to the number of arcs ξ_α in a maximal set of arcs in $\mathcal{E}$ every linear combination ξ of which has $J(\xi) \leq 0$ but no proper linear combination of which is a transversal arc.

This follows from Theorems 3:5 and 3:7.

4. Sub-problems. In this section we shall let P be a problem determined by a functional $J(\gamma)$ and a set $\mathcal{E}$ of admissible arcs defined as in Section 2. If P^* is a problem determined by the functional $J(\gamma)$ and a set of admissible arcs $\mathcal{E}^*$ where $\mathcal{E}^*$ is a linear subset of $\mathcal{E}$, then P^* is said to be a sub-problem of the problem P . Sub-problems P^* may be obtained, for example, by adding a set of end conditions to the conditions for the problem P . As an illustration consider the fixed end point problem determined by $J(\gamma)$ and a set of arcs vanishing at x_1 and x_2 . It may be obtained from the problem P by adding conditions of the form $\gamma_1(x_1) = \gamma_1(x_2) = 0$. We shall denote the fixed end point problem by P^0 and the set of arcs vanishing at x_1 and x_2 by $\mathcal{E}^0$. Sub-problems P^* of P may also be obtained by adding a set of natural isoperimetric conditions to the conditions for P , or by

defining $\mathcal{E}^*$ to be the set of all arcs in $\mathcal{E}$ which satisfy the conditions $A_{1s} \gamma_1(x^s) = 0$ ($s = 1, \dots, r < n$) where the points x^s are on the interval $x_1 x_2$. In particular we have the sub-problem determined by the set of arcs in $\mathcal{E}$ which satisfy the conditions $\gamma_1(x^s) = 0$. Unless otherwise stated we shall consider the sub-problem P^* to be such that the end conditions $\Psi_\mu = 0$ ($\mu = 1, \dots, p$) for P are linearly dependent on the end conditions $\Psi_\mu^* = 0$ ($\mu = 1, \dots, p^*$) for P^* , that is, a sub-problem obtained by adding a set of end conditions to those for P .

Before considering sub-problems further we establish the following result:

THEOREM 4:1. A necessary and sufficient condition that the end points of an arc γ in $\mathcal{E}$ can be joined by an extremal arc $\bar{\gamma}$ is that $J(\gamma, \bar{\gamma}) = 0$ for all extremal arcs $\bar{\gamma}$ in $\mathcal{E}^*$, that is, for all extremal arcs vanishing at x_1 and x_2 .

The condition is necessary, for let $\bar{\gamma}$ be an extremal arc joining the end points of γ . Then we have

$$0 = J(\gamma - \bar{\gamma}, \bar{\gamma}) = J(\gamma, \bar{\gamma}) - J(\bar{\gamma}, \bar{\gamma}) = J(\gamma, \bar{\gamma})$$

for any extremal arc $\bar{\gamma}$ in $\mathcal{E}^*$.

In order to show that the condition is sufficient let $\bar{\gamma}_\alpha$ ($\alpha = 1, \dots, k$) and a set of functions $\bar{\gamma}_\alpha = \bigcap_{\bar{\gamma}_\alpha}$, form a maximal set of linearly independent extremals in $\mathcal{E}^*$. Then we have

$$(4:1) \quad J(\bar{\gamma}_\alpha, \gamma) = \gamma_1(x_2) \bar{\gamma}_{1\alpha}(x_2) - \gamma_1(x_1) \bar{\gamma}_{1\alpha}(x_1) = 0.$$

It is well known that for any two extremals $\gamma, \bar{\gamma}$ and $\bar{\gamma}, \bar{\gamma}$ the expression $\gamma \bar{\gamma} - \bar{\gamma} \gamma$ is a constant and hence has the same value at x_1 as at x_2 (II, 738). As a consequence of this fact and the relations $\bar{\gamma}_\alpha(x_1) = \bar{\gamma}_\alpha(x_2) = 0$ it follows that the condition (4:1) holds when the values $\gamma_1(x_1), \gamma_1(x_2)$ are the end values of

an extremal arc η . Let f_γ, f_γ ($\gamma = k+1, \dots, 2n$) be a set of $(2n-k)$ extremals such that the set f_ρ, f_ρ ($\rho = 1, \dots, 2n$) form a maximal set of linearly independent extremals. Since the extremals f_α, f_α ($\alpha = 1, \dots, k$) form a maximal set of linearly independent extremals vanishing at x_1 and x_2 , the matrix

$$\begin{vmatrix} f_{1\gamma}(x_2) \\ f_{1\gamma}(x_1) \end{vmatrix} \quad (\gamma = k+1, \dots, 2n)$$

must have rank $2n-k$. The columns of this matrix accordingly form a set of $2n-k$ linearly independent solutions $\eta_1(x_2), \eta_1(x_1)$ of the equations (4:1). Moreover the equations (4:1) are linearly independent since otherwise the extremals f_α, f_α could not be linearly independent. It follows that the end values $\eta_1(x_1), \eta_1(x_2)$ of an arc η satisfying equations (4:1) are expressible in the form $\eta_1(x_1) = f_{1\gamma}(x_1)c_\gamma, \eta_1(x_2) = f_{1\gamma}(x_2)c_\gamma$ where the c 's are constants and hence the theorem is established.

COROLLARY 1. A necessary and sufficient condition that the end points of an arc η in $\mathcal{E}$ can be joined by an extremal arc $\bar{\eta}$ is that $J(\eta, f_\alpha) = 0$ where the f_α form a maximal set of linearly independent extremal arcs in $\mathcal{E}^\circ$ no proper linear combination of which is a transversal arc for P .

This follows readily from Theorem 4:1 since the condition $J(\eta, f) = 0$ is surely satisfied for all arcs f in $\mathcal{E}^\circ$ which are transversal arcs for P .

COROLLARY 2. A necessary and sufficient condition that an arc η in $\mathcal{E}$ be expressible in the form $\eta = \eta_1 + \eta_2$, where η_1 is an extremal arc in $\mathcal{E}$ and η_2 is an arc in $\mathcal{E}^\circ$, is that $J(\eta, f) = 0$ for all extremal arcs f in $\mathcal{E}^\circ$. Moreover $J(\eta) = J(\eta_1) + J(\eta_2)$ in this case.

As a consequence of Theorem 4:1 we have

THEOREM 4:2. Let γ be any arc in $\mathcal{E}$ whose end points cannot be joined by an extremal arc. There exists an arc $\bar{\gamma}$ in $\mathcal{E}$ joining the end points of γ such that $J(\bar{\gamma}) < 0$.

For, if the end points of γ cannot be joined by an extremal arc there exists an extremal arc $\bar{\gamma}$ in $\mathcal{E}^*$ such that $J(\gamma, \bar{\gamma}) \neq 0$. The arc $\bar{\gamma} = \gamma + b\bar{\gamma}$ has the same end values as γ and

$$\begin{aligned} J(\bar{\gamma}) &= J(\gamma + b\bar{\gamma}) = J(\gamma) + 2bJ(\gamma, \bar{\gamma}) + b^2 J(\bar{\gamma}) \\ &= J(\gamma) + 2bJ(\gamma, \bar{\gamma}). \end{aligned}$$

By a proper choice of b the last quantity can be made negative and the theorem is proved.

THEOREM 4:3. Let $\bar{f}_\alpha$ ($\alpha = 1, \dots, r_1$) be a maximal set of extremal arcs in $\mathcal{E}^*$ no proper linear combination of which is a transversal arc for P . Let $\bar{z}_\beta$ ($\beta = 1, \dots, r_2$) be a maximal set of arcs in $\mathcal{E}$ the end points of no linear combination of which can be joined by an extremal arc. Then $r_1 = r_2$.

For if $r_2 > r_1$ there exist constants a_β such that $J(\bar{f}_\alpha, \bar{z}_\beta a_\beta) = 0$. By Corollary 1 of Theorem 4:1 the arc $\bar{\gamma} = a_\beta \bar{z}_\beta$ can be joined by an extremal arc which is a contradiction. If $r_2 < r_1$ there exist constants b_α such that $J(b_\alpha \bar{f}_\alpha, \bar{z}_\beta) = 0$. It follows that $J(b_\alpha \bar{f}_\alpha, \gamma) = 0$ for every arc γ in $\mathcal{E}$. Hence by Theorem 3:1 the arc $\bar{\gamma} = b_\alpha \bar{f}_\alpha$ is a transversal arc contrary to our assumption. Therefore $r_1 = r_2$.

Concerning sub-problems we have several useful theorems one of which is the following:

THEOREM 4:4. A necessary and sufficient condition that an arc $\bar{\gamma}$ in $\mathcal{E}$ be expressible in the form

$$(4:2) \quad \xi = \xi_1 + \xi_2$$

where ξ_1 is an arc in $\mathcal{E}^0$ and ξ_2 is a transversal arc for P^* is that $J(\xi, \eta) = 0$ for all extremal arcs η in $\mathcal{E}^*$. Moreover $J(\xi) = J(\xi_1) + J(\xi_2)$.

The condition is necessary since the equations

$$J(\xi, \eta) = J(\xi_1 + \xi_2, \eta) = J(\xi_2, \eta)$$

hold for every extremal arc η in $\mathcal{E}^*$. By Theorem 3:1 it follows that $J(\xi_2, \eta) = 0$. We first show that the condition is sufficient when ξ is an extremal arc in $\mathcal{E}$ satisfying $J(\xi, \eta) = 0$ for all extremal arcs η in $\mathcal{E}^*$. In order to do this let ξ_α ($\alpha = 1, \dots, r$) be a maximal set of extremal arcs in $\mathcal{E}^0$ no proper linear combination of which satisfies the transversality conditions for P^* .

Let η_r ($r = 1, \dots, r$) be a set of arcs in $\mathcal{E}^*$ such that the determinant $|J(\xi_\alpha, \eta_r)|$ is different from zero. Such a set η_r exists by virtue of Theorem 4:3. Hence there exist constants a_α such that

$$(4:3) \quad J(\xi - a_\alpha \xi_\alpha, \eta_r) = J(\xi, \eta_r) - a_\alpha J(\xi_\alpha, \eta_r) = 0.$$

We now show that $\xi - a_\alpha \xi_\alpha$ is a transversal arc for P^* . Let η be any arc in $\mathcal{E}^*$ and let b_r be constants such that

$$(4:4) \quad J(\xi_\alpha, \eta - b_r \eta_r) = J(\xi_\alpha, \eta) - b_r J(\xi_\alpha, \eta_r) = 0.$$

It follows from Corollary 1 of Theorem 4:1 that the arc $\eta - b_r \eta_r$ can be joined by an extremal arc $\bar{\eta}$. Since the arcs $\eta - b_r \eta_r$, $\bar{\eta}$ have the same end values and ξ is an extremal arc we have $J(\xi, \eta - b_r \eta_r) = J(\xi, \bar{\eta})$. Moreover $J(\xi, \bar{\eta}) = 0$ as a consequence of our assumption on ξ since $\bar{\eta}$ is an extremal arc in $\mathcal{E}^*$. Consequently $J(\xi, \eta - b_r \eta_r) = 0$. By the use of this equation

and equations (4:4) it is found that

$$J(\xi - a_\alpha \xi_\alpha, \eta - b_r \eta_r) = J(\xi, \eta - b_r \eta_r) - a_\alpha J(\xi, \eta - b_r \eta_r) = 0.$$

On the other hand

$$0 = J(\xi - a_\alpha \xi_\alpha, \eta - b_r \eta_r) = J(\xi - a_\alpha \xi_\alpha, \eta) - J(\xi - a_\alpha \xi_\alpha, \eta_r) b_r.$$

It follows from this relation and equations (4:3) that

$J(\xi - a_\alpha \xi_\alpha, \eta) = 0$. Since η is any arc in $\mathcal{E}^*$ we see by the use of Theorem 3:1 that the arc $\xi - a_\alpha \xi_\alpha$ is a transversal arc for P^* . The arcs $\xi_1 = a_\alpha \xi_\alpha$, $\xi_2 = \xi - \xi_1$ are therefore related to ξ as described in the theorem. The criterion accordingly holds for an extremal arc ξ in $\mathcal{E}$.

Now suppose ξ is any arc in $\mathcal{E}$ satisfying $J(\xi, \eta) = 0$ for all extremal arcs η in $\mathcal{E}^*$. Since we have, in particular, that $J(\xi, \eta) = 0$ for all extremal arcs η in $\mathcal{E}^0$ it follows from Corollary 2 of Theorem 4:1 that ξ is expressible in the form $\xi = \eta_0 + \eta_1$ where η_0 is in $\mathcal{E}^0$ and η_1 is an extremal arc in $\mathcal{E}$. Moreover since $J(\xi, \eta) = 0$ we have $J(\eta_1, \eta) = 0$ for all extremal arcs η in $\mathcal{E}^*$. Hence by the preceding argument η_1 is expressible in the form (4:2). It follows that ξ is also expressible in this form. The last statement of the theorem is clearly true since $J(\xi_1, \xi_2) = 0$.

THEOREM 4:5. Let ξ_α ($\alpha = 1, \dots, s_1$) be a maximal set of arcs in $\mathcal{E}$ no proper linear combination of which is expressible in the form (4:2). Let η_β ($\beta = 1, \dots, s_2$) be a maximal set of extremal arcs in $\mathcal{E}^*$ no proper linear combination of which is a transversal arc for P . Then $s_1 = s_2$.

For, if $s_1 > s_2$ there exist constants a_α such that $J(\xi_\alpha a_\alpha, \eta_\beta) = 0$. Since $J(\xi_\alpha a_\alpha, \eta) = 0$ for all transversal arcs η for P it follows by Theorem 4:4 that the arc $\xi = \xi_\alpha a_\alpha$ is expressible in the form (4:2) which is a contradiction. If

$s_1 < s_2$ there exist constants b_β such that $J(\bar{f}_\alpha, \zeta_\beta b_\beta) = 0$. The arc $\gamma = b_\beta \zeta_\beta$ then satisfies the condition $J(\gamma, \bar{f}) = 0$ for all arcs $\bar{f}$ in $\mathcal{E}$. Hence γ is a transversal arc for P which is impossible. It follows that $s_1 = s_2$.

THEOREM 4:6. Let $\bar{f}_\alpha$ ($\alpha = 1, \dots, s_1$) be a maximal set of extremal arcs in $\mathcal{E}$ no proper linear combination of which is expressible in the form (4:2). Let ζ_β ($\beta = 1, \dots, s_2$) be a maximal set of extremal arcs in $\mathcal{E}^*$ no proper linear combination of which is expressible in the form $\gamma = \gamma_1 + \gamma_2$ where γ_1 is an arc in $\mathcal{E}^0$ and γ_2 is a transversal arc for P . Then $s_1 = s_2$.

The number s_1 here is not necessarily the same as the number s_1 in Theorem 4:5.

As a generalization of Corollary 2 of Theorem 4:1 we have

THEOREM 4:7. A necessary and sufficient condition that an arc $\bar{f}$ in $\mathcal{E}$ be expressible in the form

$$(4:5) \quad \bar{f} = \bar{f}_1 + \bar{f}_2$$

where $\bar{f}_1$ is an arc in $\mathcal{E}^*$ and $\bar{f}_2$ is a transversal arc for P^* is that $J(\bar{f}, \gamma) = 0$ for all transversal arcs γ for P^* in $\mathcal{E}^*$. Moreover $J(\bar{f}) = J(\bar{f}_1) + J(\bar{f}_2)$.

The condition is necessary, for, as a consequence of Theorem 3:1 we have $J(\bar{f}_1, \gamma) = 0$, $J(\bar{f}_2, \gamma) = 0$ and hence $J(\bar{f}, \gamma) = 0$ for all transversal arcs γ for P^* in $\mathcal{E}^*$. As in the proof of Theorem 4:4 we first show that the condition is sufficient for an extremal arc $\bar{f}$ in $\mathcal{E}$. To do this let ζ_α ($\alpha = 1, \dots, s$) be a maximal set of extremal arcs in $\mathcal{E}^*$ such that the determinant $|J(\zeta_\alpha, \zeta_\beta)|$ is different from zero. It follows that if $\bar{f}$ is any extremal arc in $\mathcal{E}^*$ and $J(\zeta_\alpha, \bar{f}) = 0$ then $J(\bar{f}) = 0$. Let γ be any extremal arc in $\mathcal{E}^*$ and let a_β be a set of constants such that

$$(4:6) \quad J(\gamma - a_\beta \gamma_\beta, \gamma_\alpha) = J(\gamma, \gamma_\alpha) - a_\beta J(\gamma_\beta, \gamma_\alpha) = 0.$$

The extremal arc $u = \gamma - a_\beta \gamma_\beta$ then satisfies $J(\gamma_\alpha, u) = 0$ and hence $J(u) = 0$, as was seen above. We can now show that $J(u, \bar{\gamma}) = 0$ for every extremal arc $\bar{\gamma}$ in $\mathcal{E}^*$. For, if $J(\gamma_\alpha, \bar{\gamma}) = 0$ then $J(\gamma_\alpha, u + \bar{\gamma}) = 0$ and hence $J(u + \bar{\gamma}) = 0$. Since

$$J(u + \bar{\gamma}) = J(u) + 2J(u, \bar{\gamma}) + J(\bar{\gamma}) = 2J(u, \bar{\gamma}),$$

it follows that $J(u, \bar{\gamma}) = 0$ as was to be proved. If $J(\gamma_\alpha, \bar{\gamma}) \neq 0$ then there exist constants b_β such that $J(\gamma_\alpha, \bar{\gamma} - b_\beta \gamma_\beta) = 0$ ($\beta = 1, \dots, s$). Hence $J(\bar{\gamma} - b_\beta \gamma_\beta) = 0$. Also we have the conditions $J(\gamma_\alpha, u + \bar{\gamma} - b_\beta \gamma_\beta) = 0$. It follows then that

$$0 = J(u + \bar{\gamma} - b_\beta \gamma_\beta) = 2J(u, \bar{\gamma} - b_\beta \gamma_\beta) = 2J(u, \bar{\gamma}).$$

As a consequence of Theorem 4:4 there exists an extremal arc γ_0 in $\mathcal{E}^0$ such that the arc $u - \gamma_0$ is a transversal arc for P^* .

Hence by our assumption on the extremal arc f we have $J(f, u - \gamma_0) = 0$. Clearly $J(f, \gamma_0) = 0$ and consequently $J(f, u) = 0$. Since there exist constants c_β such that

$$J(f - c_\beta \gamma_\beta, \gamma_\alpha) = J(f, \gamma_\alpha) - c_\beta J(\gamma_\beta, \gamma_\alpha) = 0,$$

it follows that

$$J(f - c_\beta \gamma_\beta, \gamma) = J(f - c_\beta \gamma_\beta, \gamma - a_\beta \gamma_\beta) = J(f - c_\beta \gamma_\beta, u),$$

the last equality holding because $\gamma - a_\beta \gamma_\beta = u$. On the other hand we have that

$$J(f - c_\beta \gamma_\beta, u) = J(f, u) - c_\beta J(\gamma_\beta, u) = 0$$

by virtue of equation (4:6) and $J(f, u) = 0$. From these relations it is seen that $J(f - c_\beta \gamma_\beta, \gamma) = 0$. Since γ is any extremal

arc in $\mathcal{E}^*$, it follows from Theorem 4:4 that there exists an extremal arc f_0 in $\mathcal{E}^0$ such that the arc $f - c_\beta \zeta_\beta - f_0$ is a transversal arc for P^* . The arcs $f_1 = a_\beta \zeta_\beta + f_0$ and $f_2 = f - f_1$ are related to f as described in the theorem. It follows that the condition of the theorem is sufficient for an extremal arc in $\mathcal{E}$.

Now suppose f is any arc in $\mathcal{E}$ and $J(f, \gamma) = 0$ for every transversal arc γ for P^* in $\mathcal{E}^*$. If the end points of f can be joined by an extremal arc γ_1 then $f = \gamma_0 + \gamma_1$ where γ_0 is an arc in $\mathcal{E}^0$. Also $J(f, \gamma) = J(\gamma_1, \gamma) = 0$ for all transversal arcs γ for P^* in $\mathcal{E}^*$. Hence by the preceding argument γ_1 is expressible in the form (4:5). It follows that f is also expressible in this form. If the end points of f cannot be joined by an extremal arc let f_α ($\alpha = 1, \dots, t$) be a maximal set of extremal arcs in $\mathcal{E}^0$ no proper linear combination of which is a transversal arc for P^* . As a consequence of Theorem 4:3 there exists a set of arcs ζ_β ($\beta = 1, \dots, t$) in $\mathcal{E}^*$ such that the determinant $|J(f_\alpha, \zeta_\beta)|$ is different from zero. Hence there exist constants d_β such that $J(f, f_\alpha) - d_\beta J(\zeta_\beta, f_\alpha) = 0$. It follows that the arc $u_1 = f - d_\beta \zeta_\beta$ can be joined by an extremal arc u_2 and $u_1 = u_0 + u_2$ where u_0 is an arc in $\mathcal{E}^0$. Also

$$J(u_2, \gamma) = J(u_1, \gamma) = J(f - d_\beta \zeta_\beta, \gamma) = J(f, \gamma) = 0$$

for all transversal arcs γ for P^* in $\mathcal{E}^*$. Again by the preceding argument u_2 is expressible in the form $u_2 = v_1 + v_2$, where v_1 is in $\mathcal{E}^*$ and v_2 is a transversal arc for P . It follows that

$f = f_1 + f_2$ where $f_1 = u_0 + v_1 + d_\beta \zeta_\beta$ is in $\mathcal{E}^*$ and $f_2 = v_2$ is the transversal arc for P^* , as was to be proved. The last statement of the theorem follows readily since $J(f_1, f_2) = 0$.

THEOREM 4:8. Let γ_α ($\alpha = 1, \dots, s_1$) be a maximal set of arcs in $\mathcal{E}$ no proper linear combination of which is expressible in the form (4:4). Let γ_ρ ($\rho = 1, \dots, s_2$) be a maximal set of transversal arcs for P^* in $\mathcal{E}^*$ no proper linear combination of which is a transversal arc for P . Then $s_1 = s_2$.

5. The index of a problem with respect to a sub-problem.
 Let P be a problem and P^* a sub-problem of P of the type studied in Section 4. Denote the indices of P and P^* by k and k^* , respectively, where by the index of P is meant the index of $J(\gamma)$ on $\mathcal{E}$. Similarly the index of P^* is the index of $J(\gamma)$ on $\mathcal{E}^*$. The difference $q^* = k - k^*$ will be called the index of P with respect to P^* . In special instances Birkhoff and Hestenes (I) and Morse (VIII) have called a number equivalent to the difference q^* the order of concavity. In the case of periodic extremals the index of repletion as defined by Morse and Pitcher (X) may be interpreted as the index of a problem with respect to a sub-problem.

LEMMA 5:1. Let γ_ρ ($\rho = 1, \dots, g$) be a maximal set of extremal arcs in $\mathcal{E}$ every proper linear combination γ of which has $J(\gamma) < 0$. Let γ_δ ($\delta = g+1, \dots, g+h$) be a maximal set of extremal arcs in $\mathcal{E}^0$ no proper linear combination of which is a transversal arc for P . Then the index q^0 of P with respect to P^0 is given by the formula $q^0 = g + h$.

In order to prove this we show that $k = k^0 + g + h$, where k and k^0 are the indices of P and P^0 , respectively. Let

ξ_ρ ($\rho = 1, \dots, k^0$) be a maximal set of arcs belonging to $\mathcal{E}^0$ every proper linear combination γ of which has $J(\gamma) < 0$. It is now sufficient to show that the conditions

$$J(\xi_\rho, \gamma) = 0 \quad (\rho = 1, \dots, k^0),$$

(5:1)

$$J(\gamma_\alpha, \gamma) = 0 \quad (\alpha = 1, \dots, g + h)$$

form a minimal set for P. Since every proper linear combination γ of these arcs has $J(\gamma) \leq 0$ but no proper linear combination is a transversal arc for P, it follows as a consequence of Theorem 3:5 that none of the conditions (5:1) can be discarded. In order to show that there are enough conditions let γ be any arc in $\mathcal{E}$ satisfying the conditions (5:1). Since γ satisfies the conditions $J(\gamma_r, \gamma) = 0$ ($r = g + 1, \dots, g + h$), it follows from Corollary 1 of Theorem 4:1 that the end points of γ can be joined by an extremal arc, that is, that γ is expressible in the form $\gamma = \gamma_1 + \gamma_2$ where γ_1 is an arc in $\mathcal{E}^0$ and γ_2 is an extremal arc in $\mathcal{E}$. By virtue of the fact that γ_2 is an extremal arc $J(\xi_\rho, \gamma_2) = 0$. Moreover, since γ_2 has the same end values as γ , we have $J(\gamma_\alpha, \gamma_2) = 0$. Hence γ_2 satisfies the conditions (5:1). It follows then that γ_1 also satisfies these conditions. As a consequence of our choice of the ξ_ρ and the γ_α we have $J(\gamma_1) \geq 0$ and $J(\gamma_2) \geq 0$. Hence $J(\gamma) \geq 0$, and the theorem follows immediately.

By means of this lemma we can prove the following significant result:

THEOREM 5:1. The index q^* of P with respect to P^* is equal to the number of extremal arcs γ_δ ($\delta = 1, \dots, s$) in a maximal set of arcs in $\mathcal{E}$ every linear combination γ of which is a transversal arc for P^* and has $J(\gamma) \leq 0$ but no proper linear combination of which is a transversal arc for P.

To prove this it is sufficient to show that $k = k^* + s$ where k and k^* are the indices of P and P^* , respectively. Consider

now the conditions

$$(5:2) \quad \begin{aligned} J(\xi_{\rho}, \gamma) &= 0 & (\rho = 1, \dots, k^*), \\ J(\gamma_s, \gamma) &= 0 & (s = 1, \dots, s), \end{aligned}$$

the first k^* of which form a minimal set for P^* . By Lemma 5:1 we may replace these k^* conditions by a set of the form (5:1) defined now for the problem P^* . Assume that this replacement has been made. We now wish to show that the $k^* + s$ conditions thus defined form a minimal set for P . As a consequence of Theorem 3:1 applied to the problem P^* the extremal arcs γ_s satisfy the conditions $J(\xi_{\rho}, \gamma_s) = 0$. With the help of this fact we have that every linear combination $\bar{\gamma}$ of the arcs ξ_{ρ}, γ_s has $J(\bar{\gamma}) \leq 0$. Since no proper linear combination of these arcs is a transversal arc for P , it now follows that none of the conditions (5:2) can be dropped. In order to show that there are enough conditions let γ be any arc in $\mathcal{E}$ satisfying the conditions (5:2). Clearly $J(u, \gamma) = 0$ for all transversal arcs u for P^* in $\mathcal{E}^*$ every linear combination of which is a transversal arc for P . Since $J(\gamma_s, \gamma) = 0$, we have that γ satisfies the condition $J(f, \gamma) = 0$ for all transversal arcs f for P^* in $\mathcal{E}^*$. Hence by Theorem 4:7 the arc γ is expressible in the form $\gamma = \gamma_1 + \gamma_2$ where γ_1 is an arc in $\mathcal{E}^*$ and γ_2 is a transversal arc for P^* . As a consequence of Theorem 3:1 we have $J(\gamma_s, \gamma_1) = 0$, $J(\xi_{\rho}, \gamma_2) = 0$. It follows then that $J(\gamma_s, \gamma_2) = 0$, $J(\xi_{\rho}, \gamma_1) = 0$. By virtue of our choice of the arcs ξ_{ρ}, γ_s we have $J(\gamma_1) \geq 0$, $J(\gamma_2) \geq 0$. Since $J(\gamma) = J(\gamma_1) + J(\gamma_2)$ it follows that $J(\gamma) \geq 0$.

COROLLARY 1. The index q^0 of P with respect to the fixed end point problem P^0 is equal to the number of extremal arcs γ_{α} ($\alpha = 1, \dots, q^0$) in a maximal set of arcs in $\mathcal{E}$ every linear combination γ of which has $J(\gamma) \leq 0$ but no proper linear

combination of which is a transversal arc for P.

COROLLARY 2. A necessary and sufficient condition that the index of P be equal to zero is that the index of P^0 be equal to zero and that the inequality $J(\gamma) \geq 0$ hold for all extremal arcs γ in $\mathcal{E}$, the equality holding only in case γ is a transversal arc for P.

COROLLARY 3. If there exist no extremal arcs in $\mathcal{E}^0$ then the index q^0 of P with respect to P^0 is equal to the number of extremal arcs γ_α ($\alpha = 1, \dots, q^0$) in a maximal set of arcs in $\mathcal{E}$ every proper linear combination γ of which has $J(\gamma) < 0$.

Consider again a problem P and a sub-problem P^* of the type already studied. We then have

LEMMA 5:2. Let p, p^* be the number of linearly independent end conditions for the problems P, P^* , respectively. Then the index q^* of P with respect to P^* satisfies the inequality

$$q^* \leq p^* - p.$$

For, let f_α ($\alpha = 1, \dots, k^*$) be a set of arcs in $\mathcal{E}^*$ such that the conditions $J(f_\alpha, \gamma) = 0$ ($\alpha = 1, \dots, k^*$) form a proper minimal set for P^* . Since P^* is a sub-problem of P, any arc satisfying the end conditions for P^* also satisfies the end conditions for P. Let γ_ν ($\nu = 1, \dots, q^*$) be a maximal set of arcs in $\mathcal{E}$ satisfying the conditions $J(f_\alpha, \gamma) = 0$ every proper linear combination γ of which has $J(\gamma) < 0$. Since no proper linear combination of the arcs γ_ν is in $\mathcal{E}^*$ and therefore not in $\mathcal{E}^0$, it follows that no two of the arcs γ_ν have the same end values. Hence $q^* \leq (2n - p) - (2n - p^*) = p^* - p$.

COROLLARY. The index q^0 of P with respect to P^0 satisfies the inequality

$$q^0 \leq 2n - p.$$

For, in this case the number corresponding to p^* of Lemma 5:2 is equal to $2n$.

It can be shown that the definition of the index of a problem with respect to a sub-problem as given by Birkhoff and Hestenes (I, 225) is not properly stated. If the extremal arcs γ_α ($\alpha = 1, \dots, h$) used by Birkhoff and Hestenes in the definition are required to be transversal arcs for the problem P^* , then their definition is equivalent to the one here given. The index of a problem with respect to a sub-problem was called, by Birkhoff and Hestenes, the order of concavity of a problem with respect to a sub-problem.

We now consider another type of sub-problem which will be of use in the following section. Let P be a problem of the type defined in Section 4 and let P^* be a sub-problem obtained from P by adding the conditions

$$(5:3) \quad \int_{x_1}^{x_2} \gamma_{1\alpha} \gamma_1 dx = 0 \quad (\alpha = 1, \dots, s),$$

where the arcs γ_α ($\alpha = 1, \dots, s$) form a maximal set of linearly independent transversal arcs for P . The class of arcs in $\mathcal{E}$ which satisfy the conditions (5:3) shall be denoted by $\mathcal{E}^*$. We then have the following result:

THEOREM 5:2. The index of P^* , where P^* is the sub-problem defined above, is equal to the index of P . Moreover there exist no transversal arcs for P^* in $\mathcal{E}^*$ except the arc $(\gamma) \equiv (0)$.

In order to prove the first statement of the theorem suppose the index of P is k and let the conditions $J(\xi_\gamma, \gamma) = 0$ ($\gamma = 1, \dots, k$) form a proper minimal set for P . If the arcs ξ_γ are in $\mathcal{E}^*$ the theorem is obviously true for then the conditions $J(\xi_\gamma, \gamma) = 0$ also form a proper minimal set for P^* . If the arcs ξ_γ are not in $\mathcal{E}^*$ then there exist constants a_γ such that the

relations

$$\int_{x_1}^{x_2} \zeta_\beta (a_{\alpha\gamma} \zeta_\alpha + f_\gamma) dx = 0 \quad (\alpha, \beta = 1, \dots, s; \gamma = 1, \dots, k)$$

hold, that is, the arcs $a_{\alpha\gamma} \zeta_\alpha + f_\gamma$ are in $\mathcal{E}^*$. Let $u_\gamma = a_{\alpha\gamma} \zeta_\alpha$, then it can be shown that the conditions $J(f_\gamma + u_\gamma, \zeta) = 0$ form a minimal set for P^* . For, if ζ is any arc in $\mathcal{E}$, then $J(u_\gamma, \zeta) = 0$ since the arcs ζ_α and hence u_γ are transversal arcs for P . We then have

$$J(f_\gamma + u_\gamma, \zeta) = J(f_\gamma, \zeta) + J(u_\gamma, \zeta) = J(f_\gamma, \zeta).$$

Since the conditions $J(f_\gamma, \zeta) = 0$ form a proper minimal set for P , it follows that any arc ζ in $\mathcal{E}$ satisfying these conditions has $J(\zeta) \geq 0$. Hence any arc ζ in $\mathcal{E}^*$ satisfying $J(f_\gamma + u_\gamma, \zeta) = 0$ has $J(\zeta) \geq 0$. Furthermore any arc of the form $\zeta = f + u$ where $f = f_\gamma b_\gamma$, $u = u_\gamma b_\gamma$ gives $J(\zeta)$ a negative value, for, $J(f, u) = J(u) = 0$, as is readily seen, and hence

$$J(f + u) = J(f) + 2J(f, u) + J(u) = J(f).$$

The functional $J(f)$ is less than zero by choice of the f_γ .

In order to prove the second statement of the theorem we first note that the Euler equations for P^* are of the form

$$\Omega \zeta_1 - \frac{d}{dx} \Omega \zeta'_1 = \mu_\alpha \zeta_\alpha \quad (\alpha = 1, \dots, s)$$

where the multipliers μ_α are constants (II, 699). An extremal arc then is a set $\zeta_1, \zeta'_1, \mu_\alpha$ satisfying these equations. If the arc ζ is a transversal arc for P^* in $\mathcal{E}$ then by an integration by parts it is seen that

$$0 = J(\zeta_\alpha, \zeta) = -\mu_\beta \int_{x_1}^{x_2} \zeta_{1\alpha} \zeta_{1\beta} dx \quad (\alpha, \beta = 1, \dots, s).$$

Since the determinant $\left| \int_{x_1}^{x_2} \gamma_{1\mu} \gamma_{1\beta} dx \right|$ is different from zero it follows that $\mu_\rho = 0$ ($\rho = 1, \dots, s$). Hence the transversal arcs for P^* in $\mathcal{E}$ are transversal arcs for P . Clearly the only transversal arcs in $\mathcal{E}$ satisfying the conditions (5:3), that is, the only transversal arc for P^* in $\mathcal{E}^*$ is $(\gamma) \equiv (0)$.

6. The one variable end point problem and focal intervals.

In this section we shall consider the one variable end point problem P determined by the functional $J(\gamma)$ and a set of arcs $\mathcal{E}$ consisting of those arcs in $\mathcal{D}$ satisfying the end conditions

$$(6:1) \quad \Psi_\mu = a_{\mu 1} \gamma_1(x_1) = 0, \quad \gamma_1(x_2) = 0 \quad (\mu = 1, \dots, p \leq n).$$

The matrix $\|a_{\mu 1}\|$ is assumed to be of rank p . The functional $J(\gamma)$ then takes the form

$$(6:2) \quad J(\gamma) = 2q(\gamma_1) + \int_{x_1}^{x_2} 2\omega(\gamma, \gamma') dx,$$

and the transversality conditions are given by

$$(6:3) \quad T_1(\gamma, \lambda) = q \gamma_{11} + \lambda_\mu a_{\mu 1} - \lambda_1 \gamma_1(x_1) = 0.$$

A transversal arc for P which satisfies the conditions $\Psi_\mu = 0$ will be called a focal arc.

The problem P determined by $J(\gamma)$ in the form (6:2) is equivalent to a problem determined by $J(\gamma)$ in the form

$$(6:4) \quad J(\gamma, w) = b_{h\lambda} w_h w_\lambda + \int_{x_1}^{x_2} 2\omega(\gamma, \gamma') dx \quad (h, \lambda = 1, \dots, r)$$

where the matrix $\|b_{h\lambda}\|$ is symmetric and the numbers w_h are constants. The end conditions are of the form

$$\gamma_1(x_1) = c_{1h} w_h, \quad \gamma_1(x_2) = 0 \quad (h = 1, \dots, r)$$

where the matrix $\|c_{1h}\|$ has maximum rank r , and the transversality

conditions are

$$\gamma_1(x_1)c_{1h} = b_h \gamma^w.$$

By adding the conditions $w_h' = \gamma'_{n+h} = 0$ it is seen that the problem formulated in this way is equivalent to a problem of the type P in $(n+r+1)$ -dimensional space. In general we shall consider $J(\gamma)$ as given in the form (6:2). We shall use the form (6:4) whenever it seems more convenient to do so.

An interval $x'x''$, not containing x_1 , will be called a focal interval of x_1 , if there exists a focal arc γ which is identically zero on $x'x''$, is not identically zero on an extension of $x'x''$ and is such that there exists no focal arc ξ having $\xi \equiv \gamma$ on an extension of $x'x''$ to the left of x' and $\xi \equiv 0$ on an extension of $x'x''$ to the right of x'' (cf. Reid (XII, 277)). A focal arc γ with the above properties will be said to correspond to the focal interval $x'x''$. Any point on a focal interval $x'x''$ will be called a focal point; the point x' being called, in particular, a primary focal point, and the point x'' a secondary focal point. Frequently a focal interval may consist of a single point. This is always the case when the problem is normal on every sub-interval (II, 725). The order of a focal interval $x'x''$ is defined to be the number of linearly independent focal arcs in a maximal set, every proper linear combination of which corresponds to the focal interval $x'x''$ in the sense described above. If the conditions $\Psi_\mu = 0$ are of the form $\gamma_1(x_1) = 0$ then the problem P reduces to the fixed end point problem P^0 and focal intervals (points) are called conjugate intervals (points).

It may be noted that in the definition of a focal interval $x'x''$ given above the words "left" and "right" may be interchanged. To prove this we need only observe that if there exists a focal

arc $\bar{f}$ having $\bar{f} \equiv \eta$ on an extension of $x'x''$ to the right of x'' and $\bar{f} \equiv 0$ on an extension of $x'x''$ to the left of x' , then the arc $f = \eta - \bar{f}$ is a focal arc having $f = \eta$ on an extension of $x'x''$ to the left of x' and $f \equiv 0$ on an extension of $x'x''$ to the right of x'' and conversely.

We shall assume throughout this section that the strengthened Clebsch condition holds on the interval x_1x_2 . We then have the following well-known result (IV).

THEOREM 6:1. The inequality $J(\eta) > 0$ holds for every arc η in $\mathcal{C}$, that is, the index of P is equal to zero if and only if there exist no focal points of x_1 on x_1x_2 .

A more general theorem is the following one:

THEOREM 6:2. The inequality $J(\eta) \geq 0$ holds for every arc in $\mathcal{C}$, that is, the index of P is equal to zero if and only if there exist no focal intervals of $x = x_1$ interior to x_1x_2 . Moreover, the number of linearly independent arcs in $\mathcal{C}$ having $J(\eta) = 0$ is equal to the number of focal intervals containing x_2 .

For, if there were a focal interval $x'x''$ of x_1 interior to x_1x_2 there would exist an arc in $\mathcal{C}$, namely, an arc which is a transversal arc on x_1x'' and identically zero on $x''x_2$ having $J(\eta) = 0$ but which is not a transversal arc on x_1x_2 . Hence the index of P on $\mathcal{C}$ could not be equal to zero. In order to prove the converse suppose there exist no focal intervals interior to x_1x_2 . If in addition there are no focal intervals containing x_2 , then $J(\eta) > 0$ by Theorem 6:1. Now suppose there are focal intervals containing x_2 and let x' be the first primary focal point. Consider first the case where x' is the only primary focal point. Let η_α ($\alpha = 1, \dots, t$) be a maximal set of linearly independent focal arcs vanishing on $x'x_2$ and define P^* to be the sub-problem obtained by adding the conditions

$$(6:5) \quad \int_{x_1}^{x_2} \gamma_{1\alpha} \gamma_1 dx = 0 \quad (\alpha = 1, \dots, t).$$

The problem P^* is then a sub-problem of the type discussed in the last part of Section 5. By Theorem 5:2 the only transversal arc in $\mathcal{E}$ satisfying the conditions (6:5) is $(\gamma) \equiv (0)$. Hence there exist no focal intervals of x_1 for P^* which contain x_2 . Since $\gamma_{1\alpha} \equiv 0$ on $x'x_2$, the point x_2 can be replaced in the above argument by any point on $x' \leq x \leq x_2$ without altering the conditions (6:5). It follows that no point of $x'x_2$ can belong to a focal interval of x_1 for P^* . Moreover, there can be no focal point of x_1 for P^* on $x_1 < x < x'$. For, if there were such a point, say x_3 , then there would exist a focal arc γ for P^* having $\gamma_1(x_3) = 0$ and $J(\gamma) = 0$ on x_1x_3 . But since there are no focal points of x_1 for P on x_1x_3 , we have by Theorem 6:1 that $J(\gamma) > 0$ on this interval. It follows again from Theorem 6:1 that the index of P^* is equal to zero. Hence by Theorem 5:2 the index of P is also equal to zero, that is, $J(\gamma) \geq 0$ for every arc γ in $\mathcal{E}$.

Now consider the case where there are other primary focal points on $x'x_2$ besides x' and let x'' be the first of these. As a consequence of the case already considered the index of $J(\gamma)$ is equal to zero on x_1x'' . Let P' denote the sub-problem of P obtained by adding the conditions $\int_{x_1}^{x_2} \gamma_{1\beta} \gamma_1 dx = 0$ where the arcs γ_β ($\beta = 1, \dots, t'$) form a maximal set of linearly independent focal arcs vanishing at x' . We next obtain a sub-problem P'' of P' by adding conditions $\int_{x_1}^{x_2} \gamma_{1\gamma} \gamma_1 dx = 0$ where the arcs γ_γ ($\gamma = t'+1, \dots, t'+t''$) form a maximal set of linearly independent arcs vanishing at x'' . We then show by the methods used before that the index of P'' is zero on x_1x''' where x''' is the

third primary focal point of x_1 . It follows that the index of P is also zero on this interval. We can continue to form sub-problems in this way. Since, in this case, the number of primary focal points of x_1 on x_1x_2 is at most n it follows that we finally obtain a sub-problem the index of which is equal to zero on the entire interval x_1x_2 . The index of P then is also zero on x_1x_2 . In fact the index remains equal to zero to the first secondary focal point. The last statement of the theorem is clearly true.

Consider now the expression

$$(6:6) \quad J(\gamma, a; x) = \gamma_{1\sigma}(x) \gamma_{1\tau}(x) a_\sigma a_\tau + \int_x^{x_2} 2\omega(\gamma, \gamma') dx$$

which is a functional of the form (6:4). The arcs γ_σ are taken to be a maximal set of focal arcs no linear combination of which vanishes at the point x . The range of σ and τ depends then upon x . This functional determines a problem P_x having the end conditions

$$(6:7) \quad \gamma_1(x) = \gamma_{1\sigma}(x) a_\sigma, \quad \gamma_1(x_2) = 0$$

and the transversality conditions

$$(6:8) \quad \gamma_1(x) \gamma_{1\sigma}(x) = \gamma_{1\sigma}(x) \gamma_{1\tau}(x) a_\tau.$$

As a consequence of these conditions we have the result:

LEMMA 6:1. An interval interior to xx_2 is a focal interval for P_x if and only if it is a focal interval for P .

For, suppose a focal interval interior to x_1x_2 is determined for the problem P by a focal arc γ . This arc γ then satisfies the first set of conditions (6:7). Hence the focal interval thus determined is also a focal interval for P_x . Conversely, an arc γ determining a focal interval for P_x must satisfy

the first of conditions (6:7) and the conditions (6:8). It follows that γ also determines a focal interval for P.

Let x' be any point on x_1x_2 . We then have the following useful lemmas.

LEMMA 6:2. If γ is any arc in $\mathcal{E}$ and if $J(\gamma, f) = 0$ for all arcs f in $\mathcal{E}$ which are identically zero on x'_1x_2 then $J(\gamma) = J(\gamma, a; x')$ where $J(\gamma, a; x')$ is defined in (6:6).

For, by Theorem 3:1 the arc γ is a transversal arc on x_1x' . Hence by an integration by parts

$$J(\gamma) = \gamma_1(x') \int_1(x') + \int_{x'}^{x_2} 2\omega(\gamma, \gamma') dx.$$

If γ_σ ($\sigma = 1, \dots, r'$) is a maximal set of linearly independent focal arcs not vanishing at x' then

$$J(\gamma) = \gamma_{1\sigma}(x') \int_{1\sigma}(x') a_\sigma a_\sigma + \int_{x'}^{x_2} 2\omega(\gamma, \gamma') dx = J(\gamma, a; x') \quad (\sigma, \tau = 1, \dots, t').$$

LEMMA 6:3. If γ is any arc in $\mathcal{E}$ and if $J(\gamma, f) = 0$ for all arcs which are focal arcs on x_1x' and identically zero on x'_1x_2 then

$$J(\gamma) = J(\gamma, a; x') + J(\gamma_1),$$

where γ_1 is an arc in $\mathcal{E}$ identically zero on x'_1x_2 .

For, by Theorem 4:7 the arc γ on the interval x_1x' is expressible in the form $\gamma = \gamma_1 + \gamma_2$ where γ_1 is an arc in $\mathcal{E}$ vanishing at x_1 and γ_2 is a focal arc. Hence on the interval x_1x_2

$$J(\gamma) = J(\gamma_1) + \gamma_{12}(x') \int_{12}(x') + \int_{x'}^{x_2} 2\omega(\gamma, \gamma') dx.$$

As in the proof of Lemma 6:2 it can be shown that the sum of the last two terms is expressible in the form $J(\gamma, a; x')$ with a

suitable set of focal arcs γ_σ and constants a_σ .

COROLLARY. If γ is any arc in $\mathcal{E}$ satisfying the conditions of Lemma 6:3 and if there exist no focal intervals of x_1 interior to x_1x' , then $J(\gamma) \geq J(\gamma, a; x')$.

This follows from the fact that $J(\gamma_1) \geq 0$ by Theorem 6:2.

It may be noted that Lemmas 6:2 and 6:3 still hold if the arc γ is restricted only to the class $\mathcal{D}$ instead of to $\mathcal{E}$ provided we choose the arcs $\gamma_{1\sigma}$ occurring in the form (6:6) to be a maximal set of linearly independent transversal arcs not vanishing at the point x' .

When the Clebsch condition holds in the strengthened form the index of P (and hence also of P_x) is finite, as will be seen in Theorem 7:4 below. Assuming for the present the truth of this fact we can prove the following theorem:

THEOREM 6:3. The index of $J(\gamma)$ on $\mathcal{E}$ is equal to the number of focal intervals of $x = x_1$ interior to x_1x_2 where a focal interval is counted a number of times equal to its order.

In order to prove this let x' be the first primary focal point of x_1 . Denote by γ_α ($\alpha = 1, \dots, t'$) a maximal set of arcs in $\mathcal{E}$ which are identically zero on $x'x_2$, are focal arcs on x_1x' and are such that no proper linear combination of these arcs are transversal arcs. It is clear that t' is equal to the number of focal intervals having x' as their initial point, each focal interval being counted a number of times equal to its order. Consider an arc γ in $\mathcal{E}$ satisfying the conditions $J(\gamma_\alpha, \gamma) = 0$. By Lemma 6:3 the value of $J(\gamma)$ is equal to the sum $J(\gamma) = J(\gamma, a; x') + J(\gamma_1)$ where γ_1 is a suitably chosen arc in $\mathcal{E}$, identically zero on $x'x_2$ and where $J(\gamma, a; x')$ is an expression of the type (6:6), the range of σ , τ being at most $n - t'$ in this case. The focal intervals for the problem P_x ,

determined by the functional $J(\gamma, a; x')$ as described above are the focal intervals for P on $x'x_2$ which do not have x' as their initial point. Moreover the difference between the indices of P and $P_{x'}$ is equal to t' . For, let the conditions $J(u_\gamma, \gamma) = 0$ ($\gamma = 1, \dots, s'$) form a minimal set for $P_{x'}$, and let $\mathcal{E}_{x'}$ denote the class of arcs associated with $P_{x'}$. It is clear from the way in which $P_{x'}$ was constructed that the arcs in $\mathcal{E}_{x'}$ are equivalent to arcs in $\mathcal{E}$ which are focal arcs on the interval x_1x' . Hence we may consider the arcs u_γ as being arcs in $\mathcal{E}$, which are focal arcs on the interval x_1x' . We now show that the conditions $J(u_\gamma, \gamma) = 0$, $J(\xi_1, \gamma) = 0$ form a minimal set for P . Clearly none of these conditions can be discarded. Now let ξ be any arc in $\mathcal{E}$ satisfying these conditions. We wish to show that $J(\xi) \geq 0$. By Lemma 6:3 we have $J(\xi) = J(\xi, a; x') + J(\xi_1)$, where ξ_1 is an arc in $\mathcal{E}$ identically zero on $x'x_2$. Since x' is the first primary focal point of x_1 it follows from the corollary to Lemma 6:3 that $J(\xi_1) \geq 0$. Also $J(\xi, a; x') \geq 0$ for, otherwise the conditions $J(u_\gamma, \gamma) = 0$ could not form a minimal set for $P_{x'}$. Hence $J(\xi) \geq 0$.

Now suppose x'' is the first primary focal point of x' for $P_{x'}$ and construct a problem $P_{x''}$ determined by a functional $J(\gamma, a; x'')$. By an argument of the type already given it follows that the difference between the indices of $P_{x'}$ and $P_{x''}$ is equal to the number of focal intervals for x' which have x'' as their initial point. The difference between the indices of P and $P_{x''}$ is then the number of focal intervals of x_1 interior to x_1x_2 and having x' and x'' as their initial points. We continue to form problems of this type at each primary focal point. At the r -th step the index of the problem P_{x^r} which corresponds to the primary focal point x^r is less than that of the problem defined at the

preceding step by an amount equal to the number of focal intervals interior to x_1x_2 and having x^r as their initial point. Since the index of P is finite one obtains after a finite number of steps a problem P_{x^m} whose index is zero. The index of P is accordingly equal to the number of focal intervals interior to x_1x_2 and having $x^1, x^2, \dots, x^m$ as their initial points. Since the index of P_{x^m} is zero, there are no focal intervals of x_1 or of x^m interior to $x^m x_2$ by virtue of Lemma 6:1 and Theorem 6:2. Thus we have accounted for all focal intervals of x_1 on the interior of x_1x_2 . This proves the theorem.

Let P and P^0 be problems as defined in Section 4. Since P^0 is a special case of the one variable end point problem, we have

COROLLARY 1. The index of P^0 is equal to the number of conjugate intervals of x_1 interior to x_1x_2 .

COROLLARY 2. The index of P is equal to the number of conjugate intervals of x_1 interior to x_1x_2 plus the index of P with respect to P^0 .

LEMMA 6:4. Given a conjugate system $\gamma_{ik}, \int_{ik}$ there exists a functional $J(\gamma)$ of the form (6:1) and a set of end conditions of the form (6:2) for which the extremals $\gamma_1, \int_1$ satisfying the end conditions $\Psi_\mu = 0$ and the transversality conditions (6:3) are of the form $\gamma_1 = \gamma_{ik}a_k, \int_1 = \int_{ik}a_k$ where the a 's are constants not all zero.

In order to prove this suppose $n-p$ is the rank of the matrix $|| \gamma_{ik}(x_1) ||$ and let the functions γ_{ik} be chosen so that $\gamma_{1\mu}(x_1) = 0$ ($\mu = 1, \dots, p$). The end conditions are then chosen to be $\Psi_\mu = a_{\mu 1} \gamma_1 = 0$ where $a_{\mu 1} = \int_{1\mu}(x_1)$, and the quadratic form $2q$ in (6:1) is taken to be

$2q = \frac{1}{2} [\int_{k1}(x_1) + \int_{ik}(x_1)] \gamma_{11} \gamma_{k1}$. The problem formed in this

way satisfies the properties of the theorem as one can readily verify.

An interval $x'x''$ is said to be a focal interval of a conjugate system γ_{1k}, γ_{ik} if there exists an arc of the form $\gamma_1 = \gamma_{1k}^{a_k}$ identically zero on $x'x''$ but not identically zero on an extension of $x'x''$ and if there exists no arc of the form $\gamma_1 = \gamma_{1k}^{b_k}$ having $\gamma \equiv \gamma$ on an extension of $x'x''$ to the left of x' and $\gamma \equiv 0$ on an extension of $x'x''$ to the right of x'' . As in the definition of a focal interval given earlier in this section, the words "left" and "right" may also be interchanged here. A focal interval of a conjugate system on an interval x_1x_2 may also be said to be a focal interval of a one variable end point of the problem determined by the system. Let γ_{1k}, u_{1k} be the extremal arcs associated with two conjugate systems and let P_1, P_2 be the problems determined by these systems on the interval x_1x_2 . For convenience suppose the variable end point for P_1 is x_1 and the variable end point for P_2 is x_2 . If we let P^0 denote the fixed end point problem then we have the following result.

LEMMA 6:5. Let t_1 denote the number of focal intervals of x_1 and t_2 the number of focal intervals of x_2 interior to the interval x_1x_2 for the problems P_1 and P_2 respectively, where P_1 and P_2 are defined as above. Let q_1, q_2 denote respectively the indices of P_1, P_2 with respect to P^0 . Then

$$t_1 - t_2 = q_1 - q_2.$$

This follows from Corollary 2 of Theorem 6:3 since the number of conjugate intervals of x_1 is the same as the number of conjugate intervals of x_2 interior to x_1x_2 .

We now consider an example of a one variable end point problem P in a 4-dimensional space having focal intervals $x'x''$, $t't''$ lying interior to x_1x_2 with $x' < t' \leq x''$. Let $u(x)$ be an arc of class C^n such that $u(x_1) = 0$ and $u(x) \equiv 0$ on $x'x''$ but not on an extension of $x'x''$. Similarly let $v(x)$ be an arc of class C^n such that $v(x) \equiv 0$ on x_1x' , and $t't''$ but not on an extension of $t't''$. Let $h_1 = u'' + u$, $h_2 = v'' + v$ and

$$b_1 = \int_{x_1}^{x'} u h_1 dx = - \int_{x_1}^{x'} (u'^2 - u^2) dx, \quad b_2 = \int_{x_1}^{t'} v h_2 dx = - \int_{x_1}^{t'} (v'^2 - v^2) dx.$$

The problem P under consideration is determined by the functional

$$J(\gamma, w) = b_1 w_1^2 + b_2 w_2^2 + \int_{x_1}^{x_2} (\gamma_1'^2 - \gamma_1^2) dx,$$

the differential equations

$$\gamma_2' = \gamma_1 h_1, \quad \gamma_3' = \gamma_1 h_2$$

and the end conditions

$$\gamma_1(x_1) = 0, \quad \gamma_2(x_1) = -b_1 w_1, \quad \gamma_3(x_1) = -b_2 w_2, \quad \gamma_1(x_2) = 0 \\ (i = 1, 2, 3),$$

where w_1, w_2 are constants. The Euler equations and transversality conditions associated with the problem P are, respectively,

$$\gamma_1'' + \gamma_1 + \lambda_1 h_1 + \lambda_2 h_2 = 0, \quad \lambda_1' = 0, \quad \lambda_2' = 0,$$

and

$$\gamma_2(x_1) = -w_1, \quad \gamma_3(x_1) = -w_2.$$

It is readily verified that the arc γ defined by

$$(6:9) \quad \gamma_1 = u, \quad \gamma_2 = \int_{x_1}^x u h_1 dx - b_1 w_1, \quad \gamma_3 = \int_{x_1}^x u h_2 dx - b_2 w_2$$

with multipliers $\lambda_1 = -1$, $\lambda_2 = 0$ and constants $w_1 = 1$, $w_2 = 0$ is

a focal arc for P and is identically zero on $x'x''$. Similarly the arc η defined by

$$\eta_1 = v, \quad \eta_2 = \int_{x_1}^x v h_1 dx - b_1 w_1, \quad \eta_3 = \int_{x_1}^x v h_2 dx - b_2 w_2$$

with multipliers $\lambda_1 = 0$, $\lambda_2 = -1$ and constants $w_1 = 0$, $w_2 = 1$ is also a focal arc for P and is identically zero on $t't''$. If $t' > x''$ we choose the function $v(x)$ so that $\int_{x''}^{t'} v h_1 dx = 0$. It may be noted that if $x_1 = 0$ and $x' > \pi$ then the function $u(x)$ can be chosen so that $b_1 = 0$. The arc (6:9) then defines a conjugate interval on $x'x''$.

7. A sub-problem of P^0 . Let $x = x^s$ ($s = 0, \dots, r+1$; $x^0 = x_1$, $x^{r+1} = x_2$) be a set of points arranged in the order of their superscripts. In this section we shall consider the problem P^* determined by a functional of the form (1:1) and a set of arcs ξ^* in $\mathcal{D}$ which vanish at the points x^s . Clearly ξ^* is a subset of ξ^0 and P^* is a sub-problem of P^0 where ξ^0 and P^0 are defined as in Section 4. It is to be noted that the extremal arcs for P^* are broken extremal arcs for P^0 with corners at the interior points x^s ($s = 1, \dots, r$). We shall assume that the strengthened Clebsch condition holds on $x_1 x_2$.

For convenience we shall define the index of a sub-interval $x^s x^{s+1}$ ($s = 0, \dots, r$) to be the number of arcs in a maximal set of arcs in ξ^* identically zero except on the interval $x^s x^{s+1}$ ($s = 0, \dots, r$) every proper linear combination of which has $J(\eta) < 0$. The index of $x^s x^{s+1}$ ($s = 0, \dots, r$) is then equal to the number of conjugate intervals of x^s interior to this interval.

THEOREM 7:1. The index of P^* is equal to the sum of the indices of the sub-intervals $x^s x^{s+1}$ ($s = 0, \dots, r$).

For, let η be any arc in ξ^* having $J(\eta) < 0$. It is readily seen that η is equal to a sum of arcs in ξ^* each being

identically zero except on an interval $x^s x^{s+1}$ and each having $J < 0$.

Let P be the problem defined in Section 4. We then have

THEOREM 7:2. The index q^* of P with respect to P^* is equal to the number of broken extremal arcs γ_α ($\alpha = 1, \dots, q^*$) with corners at the interior points x^s ($s = 1, \dots, r$) in a maximal set of arcs in $\mathcal{E}$ every linear combination γ of which has $J(\gamma) \leq 0$ but no proper linear combination of which is a transversal arc for P .

This is the same as Theorem 5:1 stated for a different type of sub-problem and it is easily verified that the theorem holds in this case.

COROLLARY. If there exist no extremal arcs in $\mathcal{E}^*$, then the index q^* of P with respect to P^* is equal to the number of broken extremal arcs γ_α ($\alpha = 1, \dots, q^*$) with corners at x^s ($s = 1, \dots, r$) in a maximal set of arcs in $\mathcal{E}$ every proper linear combination γ of which has $J(\gamma) < 0$.

THEOREM 7:3. If the points x^s ($s = 0, \dots, r+1$) are such that there exist no conjugate points of x^s on $x^s x^{s+1}$ ($s = 0, \dots, r$), then the index of P is equal to the number k of broken extremal arcs with corners at the interior points x^s ($s = 1, \dots, r$) in a maximal set of arcs in $\mathcal{E}$ every proper linear combination γ of which has $J(\gamma) < 0$.

For, the index of each sub-interval $x^s x^{s+1}$ ($s = 0, \dots, r$) is equal to zero. Hence by Theorem 7:1 the index of P^* is equal to zero and, therefore, the index of P is the same as the index of P with respect to P^* . This number k is equivalent to the negative index of the index form of Morse (VIII).

LEMMA 7:1. A finite number r of points x^s ($s = 1, \dots, r$) can always be chosen on $x_1 x_s$ such that there exist no pairs of

conjugate points on $x^s x^{s+1}$ ($s = 0, \dots, r$).

For, suppose that the lemma is false. Choose arbitrarily a finite number of points on $x_1 x_2$. The lemma is then false on at least one of the sub-intervals thus formed. On this interval make a further subdivision. It follows that the lemma is false on at least one of these sub-intervals. In this manner continue to make subdivisions of the interval $x_1 x_2$. The points of subdivision will have a limit point x' and it follows that in every neighborhood of x' there exist pairs of conjugate points. But this contradicts the known fact that for any point x'' on $x_1 x_2$ there exists an interval $x'' - \delta \leq x \leq x'' + \delta$ on which there are no pairs of conjugate points (II, 739). Hence the lemma is true.

THEOREM 7:4. If the strengthened Clebsch condition holds on $x_1 x_2$ then the index k of P is finite.

For, by Lemma 7:1 we can choose the points x^s on $x_1 x_2$ so that there exist no conjugate points of x^s on $x^s x^{s+1}$ ($s = 1, \dots, r$). Let r be the number of sub-intervals $x^s x^{s+1}$. Then the number k of Theorem 7:3 satisfies the inequality $k \leq nr$ and hence the index of P is finite.

Let x^s ($s = 0, \dots, r+1$) be a set of points as described in the first paragraph of this section. Let t be the number of broken extremal arcs f_α ($\alpha = 1, \dots, t$) in $\bar{C}$ with corners at the interior points x^s ($s = 1, \dots, r$) every proper linear combination f of which has $J(f) < 0$ and let v be the number of linearly independent arcs g_β ($\beta = 1, \dots, v$) vanishing at x^s ($s = 0, \dots, r+1$) which are transversal arcs for P , where P is defined as in Section 4. We then have the following

THEOREM 7:5. The index k of P is expressible in the form

$$k = p_0 + \dots + p_r + t - v,$$

where p_s is the number of primary conjugate points of x^s on $x^s x^{s+1}$ ($s = 0, \dots, r$) and t and v are defined as above.

In order to prove this let γ^s ($s = 1, \dots, p_s; s = 0, \dots, r$) be a maximal set of broken extremal arcs determining the primary conjugate points of x^s ($s = 0, \dots, r$) on $x^s x^{s+1}$. These may be chosen so as to be identically zero except on $x^s x^{s+1}$. By the types of proofs used before it is readily verified that every arc γ in $\mathcal{E}$ satisfying the conditions

$$(7:1) \quad J(\gamma^s, \gamma) = 0, \quad J(\xi_\alpha, \gamma) = 0$$

has $J(\gamma) \geq 0$. Let $\bar{\gamma}^s$ ($s = p_0 + \dots + p_r - v$) be a maximal set of the arcs γ^s no proper linear combination of which is a transversal arc for P . The conditions $J(\bar{\gamma}^s, \gamma) = 0$, $J(\xi_\alpha, \gamma) = 0$ then form a minimal set, for, every arc γ in $\mathcal{E}$ satisfying these conditions also satisfies the conditions (7:1) and hence has $J(\gamma) \geq 0$. Moreover every linear combination ξ of the arcs $\bar{\gamma}^s$, ξ_α has $J(\xi) \leq 0$ but no proper linear combination is a transversal arc for P . Hence the statement of the theorem is true.

Results analogous to those of this section may be verified for sub-problems P^* of P obtained by imposing the conditions $A_{1\alpha}^s \gamma_1(x^s) = 0$ where the $A_{1\alpha}^s$ are constants and the points x^s are finite in number and lie interior to the interval $x_1 x_2$.

THEOREM 7:6. If the index k of $J(\gamma)$ on $\mathcal{E}$ is finite then the Clebsch condition holds on $x_1 x_2$.

For, since k is finite and the index q° of P with respect to P° is finite, it follows that the index k° of $J(\gamma)$ on $\mathcal{E}^\circ$ is also finite. Define the index of a sub-interval $x'x''$ of $x_1 x_2$ to be the number of arcs in a maximal set of arcs in $\mathcal{E}^\circ$ identically zero except on $x'x''$ every proper linear combination γ of which has $J(\gamma) < 0$. The index of $x'x''$ is clearly less than or equal to

the index of $J(\gamma)$ on $\mathcal{E}^0$. It follows that the interval x_1x_2 can be subdivided into a finite number of intervals of length less than δ such that on all but a finite number of these the index is equal to zero. On those intervals for which the index is zero the arc $(\gamma) \equiv (0)$ is a minimizing arc and hence the Clebsch condition holds. Since δ may be taken arbitrarily small it follows by continuity considerations that the Clebsch condition holds on the entire interval x_1x_2 .

8. Oscillation and comparison theorems. We shall assume that the strengthened Clebsch condition holds on the interval x_1x_2 . Let P and P^0 be problems as defined in Section 4. As a consequence of Corollary 2 of Theorem 6:3 we have the following general oscillation theorem.

THEOREM 8:1. Let k be the index of P and q^0 the index of P with respect to P^0 . There exist exactly $k - q^0$ conjugate intervals of x_1 or of x_2 interior to x_1x_2 .

As a consequence of the corollary to Lemma 5:2 we have

COROLLARY. The number k^0 of conjugate intervals of x_1 interior to x_1x_2 satisfies the inequality

$$k - 2n + p \leq k^0 \leq k.$$

Let P^* be a problem determined by a functional J^* and a set of end conditions $\Psi_{\tau}^* = 0$ ($\tau = 1, \dots, p^*$). We then have the following comparison theorems.

THEOREM 8:2. Let k, k^* denote the indices and s, s^* the number of conjugate intervals for x_1 interior to x_1x_2 for P and P^* , respectively. Let q, q^* denote respectively the indices of P and P^* with respect to P^0 . We then have the relations

$$k - k^* = q - q^* + s - s^*,$$

$$|k - k^*| \leq |s - s^*| + \max[2n - p, 2n - p^*].$$

The first result follows from Corollary 2 of Theorem 6:3. The second statement then follows as a consequence of the corollary of Lemma 5:2.

COROLLARY. If $\omega(\gamma, \gamma') = \omega^*(\gamma, \gamma')$, then

$$k - k^* = q - q^*,$$

$$|k - k^*| \leq \max[2n - p, 2n - p^*] \leq 2n.$$

These statements follow from Theorem 8:2 since $s - s^* = 0$.

Now suppose that P_1, P_1^* are sub-problems of P, P^* respectively and of the type studied in Section 4. Let k_1, k_1^* denote the indices and p_1, p_1^* the number of end conditions for P_1, P_1^* , respectively. We then have

THEOREM 8:3. Let q_1, q_1^* denote the indices of P, P^* with respect to P_1, P_1^* , then

$$k - k^* = q_1 - q_1^* + k_1 - k_1^*,$$

$$|k - k^*| \leq |k_1 - k_1^*| + \max[p_1 - p, p_1^* - p^*].$$

If the sub-problems P_1, P_1^* are identical, then

$$k - k^* = q_1 - q_1^*,$$

$$|k - k^*| \leq \max[p_1 - p, p_1^* - p^*] \leq 2n.$$

If the problems P, P^* are identical, then

$$k_1 - k_1^* = q_1^* - q_1,$$

$$|k_1 - k_1^*| \leq \max[p_1 - p, p_1^* - p] \leq 2n.$$

Now consider two problems P, P^* having the same set of arcs $\mathcal{C}$ and let

$$J'(\gamma) = J(\gamma) - J^*(\gamma).$$

The problem determined by $J'(\gamma)$ and the set $\mathcal{E}$ shall be denoted by $P' = P - P^*$.

THEOREM 8:4. If k' is the index of P' then

$$k \leq k^* + k'.$$

In order to prove this we shall suppose that k' is finite, for, the theorem is surely true if k' is infinite. Let the conditions

$$(8:1) \quad J'(\xi_\alpha, \gamma) = 0 \quad (\alpha = 1, \dots, k')$$

form a proper minimal set for the problem P' . It follows that for every arc γ in $\mathcal{E}$ satisfying these conditions $J(\gamma) \geq J^*(\gamma)$. Let γ_r ($r = 1, \dots, h$) be a maximal set of arcs in $\mathcal{E}$ satisfying these conditions every linear combination ξ of which has $J(\xi) < 0$. Every arc γ in $\mathcal{E}$ is expressible as a linear combination of the arcs ξ_α and an arc satisfying conditions (8:1). Clearly then $k \leq k' + h$. Since the arcs γ_r satisfy (8:1), every linear combination $\gamma = a_r \gamma_r$ has $J^*(\gamma) < 0$. Hence $h \leq k^*$ and $k \leq k^* + k'$.

THEOREM 8:5. Let k, k^* be the indices of P, P^* respectively and let d, d^* denote the maximum number of linearly independent transversal arcs in $\mathcal{E}$ for P, P^* respectively. If $J(\gamma) \geq J^*(\gamma)$ for every arc $(\gamma) \neq (0)$ in $\mathcal{E}$, then

$$k^* \geq k, \quad k^* + d^* \geq k + d.$$

Moreover, if $J(\gamma) > J^*(\gamma)$ for all such arcs, then

$$k^* \geq k + d.$$

In order to prove this let

$$J(\bar{f}_\alpha, \gamma) = 0 \quad (\alpha = 1, \dots, k)$$

be a minimal set for P and let $\bar{f}_\gamma$ ($\gamma = k+1, \dots, k+d$) be a maximal set of linearly independent transversal arcs for P in $\mathcal{E}$. We then have

$$J(\bar{f}_\alpha a_\alpha) \leq 0 \quad (\alpha = 1, \dots, k+d),$$

the inequality holding only in case the first k a 's are zero. It follows that

$$J^*(\bar{f}_\alpha a_\alpha) \leq 0 \quad (\alpha = 1, \dots, k+d),$$

the strict inequality holding in case the first k a 's are not all zero. Hence $k^* \geq k$. We now consider a maximal set $\bar{f}_\sigma$ of the $\bar{f}_\alpha$ ($\alpha = 1, \dots, k+d$) no proper linear combination of which is a transversal arc for J^* . The number of arcs in the set $\bar{f}_\sigma$ is at least $k+d-d^*$. Since these arcs form a subset of a set of functions defining a minimal set for P^* it follows that $k^* \geq k+d-d^*$. Hence the second inequality is proved. If $J(\gamma) > J^*(\gamma)$ for every arc $(\gamma) \neq (0)$ in $\mathcal{E}$ then $J^*(\bar{f}_\alpha a_\alpha) < 0$ ($\alpha = 1, \dots, k+d$) for every set of constants $(a) \neq (0)$. Hence $k^* \geq k+d$.

Now let P be the problem determined by the functional

$$J(\gamma) = 2q_1(\gamma_1) - 2q_2(\gamma_2) + \int_{x_1}^{x_2} 2\omega(\gamma, \gamma') dx$$

and a set of arcs $\mathcal{E}$, the maximal set of arcs in $\mathcal{D}$ satisfying the end conditions

$$(8:2) \quad \begin{aligned} \psi_{\mu 1}(\gamma_1) &= a_{\mu 1} \gamma_1(x_1) = 0 & (\mu = 1, \dots, p_1 \leq n), \\ \psi_{\tau 2}(\gamma_2) &= a_{\tau 1} \gamma_1(x_2) = 0 & (\tau = 1, \dots, p_2 \leq n), \end{aligned}$$

where $2q_1$ and $2q_2$ are symmetric quadratic forms in γ_{11}, γ_{12} respectively. The transversality conditions are then expressible in the form

$$\begin{aligned} T_{11}(\gamma_1, \lambda, \gamma_1) &= q_1 \gamma_{11} + \gamma_{11}^2 \mu_1 - \gamma_1(x_1) = 0, \\ T_{12}(\gamma_2, \lambda, \gamma_2) &= q_2 \gamma_{12} + \gamma_{12}^2 \mu_2 + \gamma_1(x_2) = 0. \end{aligned} \quad (8:3)$$

A sub-problem P_1 may be obtained by replacing the end conditions $\gamma_{12} = 0$ by $\gamma_1(x_2) = 0$. Also by replacing the end conditions $\gamma_{11} = 0$ by $\gamma_1(x_1) = 0$ a sub-problem P_2 may be obtained.

Let P^* be a sub-problem of the problem P defined in the preceding paragraph. Suppose P^* is determined by a set of arcs $\tilde{C}^*$ consisting of those arcs in $\tilde{C}$ vanishing at x' . Then P^* is a sub-problem of the type studied in Section 7 having variable end points instead of fixed end points. Let $q(x')$ denote the number of broken extremal arcs γ_α ($\alpha = 1, \dots, q(x')$) in a maximal set of arcs in $\tilde{C}$ having a single corner at x' every linear combination γ of which has $J(\gamma) \leq 0$ and satisfies the transversality conditions for P but no proper linear combination of which is a transversal arc for P . If t_1 is equal to the number of focal intervals of x_1 interior to $x_1 x'$ and t_2 is equal to the number of focal intervals of x_2 interior to $x' x_2$ then the index k of P is expressible in the form

$$(8:4) \quad k = t_1 + t_2 + q(x').$$

Let γ_{1k} be the extremal arcs of a conjugate system determined by the first p_1 of conditions (8:2) and (8:3) and let u_{1k} be the extremal arcs of a conjugate system determined by the last p_2 of conditions (8:2) and (8:3). Let ξ_ρ ($\rho = 1, \dots, r_1$) be a maximal set of linearly independent extremal arcs of the system γ_{1k} which vanish at x' and ξ_r ($r = r_1 + 1, \dots, r_1 + r_2 = r$) a

maximal set of linearly independent extremal arcs of the system u_{1k} which vanish at x' no linear combination of which belongs to γ_{1k} . It follows that the matrix $\|\zeta_{1\delta}(x')\|$ ($\delta = 1, \dots, r$) has maximum rank r , where the functions $\zeta_{1\delta}$ are those associated with the extremal arcs $\xi_{1\delta}$. Define $\bar{\xi}_{1\delta} \equiv \xi_{1\delta}$ on x_1x' , $\bar{\xi}_{1\delta} \equiv 0$ elsewhere; $\bar{\xi}_r \equiv \xi_r$ on $x'x_2$, $\bar{\xi}_r \equiv 0$ elsewhere. The r arcs $\bar{\xi}_r$ thus formed are then broken extremal arcs in $\mathcal{E}^*$. Since $J(\bar{\xi}_r, \gamma) = 0$ for all broken extremal arcs γ in $\mathcal{E}^*$ with corners at $x = x'$ it follows that

$$\zeta_{1\delta}(x') \gamma_1(x') = 0$$

for all such arcs γ . This system of equations has $n-r$ linearly independent non-zero solutions. Since on the interval x_1x' the arcs γ_α described above are linear combinations of the arcs γ_{1k} and on $x'x_2$ they are linear combinations of the arcs u_{1k} , it now follows that $q(x') \leq n-r+r-d = n-d$, where d is the maximum number of linearly independent transversal arcs for P no proper linear combination of which vanishes at x' . A stronger but more complicated inequality can be obtained. However, the one obtained here is sufficient for the arguments to be given in Section 10. It is clear that if x' is not a focal point of either system then d is equal to the maximum number of linearly independent transversal arcs for P .

THEOREM 8:6. If t_1 and t_2 denote the number of focal intervals of x_1 and x_2 , respectively, interior to the interval x_1x_2 , then the following relation holds:

$$t_1 - t_2 = q(x_1 + 0) - q(x_2 - 0).$$

Moreover $|t_1 - t_2| \leq n$.

For, in the expression (8:4) we let x' approach x_1 and then let x' approach x_2 obtaining the relations $k = t_2 + q(x_1 + 0)$ and $k = t_1 + q(x_2 - 0)$, respectively. The first relation of the theorem then follows immediately. The last relation is true since both $q(x_1 + 0)$ and $q(x_2 - 0)$ are less than or equal to n .

Let η_{ik} , ξ_{ik} and u_{ik} , v_{ik} be a pair of conjugate systems. We then have the following

THEOREM 8:7. If t_1 and t_2 denote respectively the number of focal intervals of the conjugate systems η_{ik} , ξ_{ik} , and u_{ik} , v_{ik} on a given interval γ then the following relations hold

$$\begin{aligned} t_1 - t_2 &= q(x_1 + 0) - q(x_2 - 0) && \text{on} && (x_1 < x < x_2), \\ t_1 - t_2 &= q(x_1 + 0) - q(x_2 + 0) && \text{on} && (x_1 < x \leq x_2), \\ t_1 - t_2 &= q(x_1 - 0) - q(x_2 - 0) && \text{on} && (x_1 \leq x < x_2), \\ t_1 - t_2 &= q(x_1 - 0) - q(x_2 + 0) && \text{on} && (x_1 \leq x \leq x_2), \end{aligned}$$

where $q(x')$ is defined as before for the problem P determined by η_{ik} and u_{ik} . In each of these cases $|t_1 - t_2| \leq n$.

The first relation follows immediately from Theorem 8:6.

In order to obtain the second relation we consider the problem P defined on the interval $x_1 x_3$ where x_3 is to the right of x_2 but sufficiently near so that the number of focal intervals of η_{ik} and u_{ik} on $x_1 x_3$ is the same as on $x_1 x_2$. By the same type of argument used to obtain the first relation we can now obtain the second. The remaining relations can be verified similarly.

9. The accessory boundary value problem. Let $J(\gamma)$ be defined as in Section 1 and let $\mathcal{E}$ be the class of arcs as defined in Section 2. We shall consider the functional

$$J(\gamma, \sigma) = J(\gamma) - \sigma \int_{x_1}^{x_2} \eta_1 \xi_1 dx$$

called the characteristic functional of $J(\gamma)$. A value of σ for which there exists a non-zero transversal arc γ_1 for the problem determined by $J(\gamma, \sigma)$ on $\mathcal{E}$ is called a characteristic value. The functions $\gamma_1(x)$ defining this transversal arc are said to form a characteristic arc. Associated with a given characteristic arc γ are a set of multipliers $\lambda_\mu(x)$, λ_ν such that the Euler equations

$$L_1(\gamma) - \sigma \gamma_1 = 0$$

and the transversality conditions

$$T_{11}(\gamma, \lambda, \lambda) = 0, \quad T_{12}(\gamma, \lambda, \lambda) = 0$$

for the problem are satisfied. The functions L_1 , T_{11} , T_{12} are defined as in equations (2:1) and (2:2).

In this section let $k(\sigma)$ denote the index of $J(\gamma, \sigma)$ on $\mathcal{E}$ and let $d(\sigma)$ denote the number of linearly independent characteristic arcs associated with σ . The number $d(\sigma)$ will be called the order of σ as a characteristic value. It is understood that if σ is not a characteristic value then $d(\sigma) = 0$.

In order to establish a relationship between the index of $J(\gamma)$ on $\mathcal{E}$ and the characteristic values of $J(\gamma, \sigma)$ we prove the following lemmas. We shall assume that the strengthened Clebsch condition holds on $x_1 x_2$.

LEMMA 9:1. The index $k(\sigma)$ of $J(\gamma, \sigma)$ on $\mathcal{E}$ is equal to zero for σ sufficiently large.

In order to prove this we first note that by the theory of quadratic forms there exists a constant c such that $2q(\gamma)$ in (1:1) satisfies the relation $2q(\gamma) \geq c(\gamma_{11}\gamma_{11} + \gamma_{12}\gamma_{12})$. Let $h(x)$ be any function of x of class C^1 on $x_1 x_2$ such that $h(x_1) = -1$, $h(x_2) = +1$. Then

$$\begin{aligned}
 J(\gamma, \sigma) &\geq \int_{x_1}^{x_2} \left[c \frac{d}{dx} (\gamma_1 \gamma_1^h) + 2\omega(x, \gamma, \gamma') - \sigma \gamma_1 \gamma_1 \right] dx \\
 (9:1) \quad &= \int_{x_1}^{x_2} [P_{1k}^* \gamma_1 \gamma_k + 2Q_{1k}^* \gamma_1 \gamma'_k + R_{1k} \gamma'_1 \gamma'_k - \sigma \gamma_1 \gamma_1] dx,
 \end{aligned}$$

where P_{1k}^*, Q_{1k}^* are functions having the properties of P_{1k}, Q_{1k} (VII, 50-51). Now consider

$$Q = P_{1k}^* \gamma_1 \gamma_k \tau^2 + 2Q_{1k}^* \gamma_1 \gamma'_k \tau + R_{1k} \gamma'_1 \gamma'_k + \gamma_1 \gamma_1,$$

where γ_1, γ'_1 satisfy

$$(9:2) \quad S_{\nu 1} \gamma_1 \tau + T_{\nu 1} \gamma'_1 = 0.$$

When τ is zero then γ'_1 satisfies $T_{\nu 1} \gamma'_1 = 0$ and $Q > 0$ since the strengthened Clebsch condition holds. By continuity considerations it follows that for functions γ_1, γ'_1 satisfying (9:2) we have $Q > 0$ for τ near zero. Now let $\tau = \gamma'_1 \gamma_1$ where γ_1 is near zero, then

$$Q = P_{1k}^* \gamma_1 \gamma_k \tau^2 + 2Q_{1k}^* \gamma_1 \gamma'_k \tau + R_{1k} \gamma'_1 \gamma'_k \tau^2 + \gamma_1 \gamma_1 > 0.$$

Hence $Q \tau^{-2} > 0$. Setting $\sigma = \tau^{-2}$ in (9:1) it follows that the integrand becomes $Q \tau^{-2}$, and hence $J(\gamma, \sigma) > 0$ for $-\sigma$ sufficiently large.

LEMMA 9:2. If σ is sufficiently near a fixed value σ_0 , then

$$k(\sigma) = k(\sigma_0) \quad (\sigma < \sigma_0), \quad k(\sigma) = k(\sigma_0) + d(\sigma_0) \quad (\sigma > \sigma_0).$$

Moreover, on an interval $\sigma_1 \leq \sigma < \sigma_2$ there are exactly $k(\sigma_2) - k(\sigma_1)$ characteristic values where a characteristic value is counted a number of times equal to its order.

For, if $\sigma < \sigma_0$ then $J(\gamma, \sigma) > J(\gamma, \sigma_0)$. Hence by Theorem 8:5 we have $k(\sigma_0) \geq k(\sigma) + d(\sigma)$. By virtue of this

inequality there can be at most a finite number of characteristic values on an interval $\sigma_1 \leq \sigma \leq \sigma_2$. Otherwise $k(\sigma)$ would become infinite as σ varies from σ_1 to σ_2 and this is impossible. Hence the characteristic values are isolated. Now let $\xi_\alpha(x)$ ($\alpha = 1, \dots, k(\sigma_0)$) be a set of arcs in $\mathcal{E}$ and having $J[\xi_\alpha, \xi_\beta, \sigma]_{a_\alpha}^{a_\beta} < 0$ ($\alpha, \beta = 1, \dots, k(\sigma_0)$) for $\sigma = \sigma_0$, where $J(\xi, \eta, \sigma)$ is a functional of the type (3:1) formed for $J(\eta, \sigma)$. Since this relation holds for σ near σ_0 it follows that $k(\sigma) \geq k(\sigma_0)$. Hence $k(\sigma) = k(\sigma_0)$. Also for σ sufficiently near σ_0 we have $d(\sigma) = 0$ since σ is not a characteristic value.

If $\sigma > \sigma_0$ then by the preceding argument $k(\sigma) \geq k(\sigma_0) + d(\sigma_0)$. In order to show that $k(\sigma) \leq k(\sigma_0) + d(\sigma_0)$ choose points x^s ($s = 0, \dots, h+1$) with $x^0 = x_1$, $x^{h+1} = x_2$ arranged in order of their superscripts on $x_1 x_2$ such that there exist no conjugate intervals of x^s on $x^s x^{s+1}$ ($s = 0, \dots, h$). Let $\gamma_r(x, \sigma)$ ($r = 1, \dots, r$) be a maximal set of r linearly independent broken extremal arcs in $\mathcal{E}$ continuous in σ and with corners at the points x^s ($s = 1, \dots, h$). The index $k(\sigma)$ and the order $d(\sigma)$ of σ as a characteristic value are equal respectively to the negative index and nullity of the quadratic form

$$J[\gamma_r(\sigma), \gamma_s(\sigma), \sigma] b_r b_s \quad (r, s = 1, \dots, r).$$

As seen above $d(\sigma) = 0$ for σ sufficiently near σ_0 . Hence as σ increases or decreases from σ_0 the index $k(\sigma)$ increases by at most $d(\sigma_0)$, that is,

$$k(\sigma) \leq k(\sigma_0) + d(\sigma_0).$$

It follows that $k(\sigma) = k(\sigma_0) + d(\sigma_0)$ if $\sigma > \sigma_0$. Let the characteristic values on $\sigma_1 \sigma_2$ be $\sigma' = \sigma_1, \sigma'', \dots, \sigma^d = \sigma_2$.

and suppose $\sigma^1 < \sigma^2 < \dots < \sigma^q$. Since $k(\sigma^s) = k(\sigma^{s-1}) + d(\sigma^{s-1})$, ($s = 2, \dots, q$), it follows that

$$k(\sigma_q) - k(\sigma_1) = d(\sigma^1) + \dots + d(\sigma^{q-1})$$

and hence the last statement of the lemma is proved. It is clear that if $k(\sigma_1) = 0$ then $k(\sigma_q)$ is equal to the number of characteristic values less than σ_q . This follows from the above formula since for $\sigma < \sigma_1$, $k(\sigma) = k(\sigma_1) = 0$ and $d(\sigma) = 0$.

THEOREM 9:1. If the strengthened Clebsch condition holds on x_1, x_2 then the index k of $J(\gamma)$ on $\mathcal{C}$ is equal to the number of negative characteristic values of its characteristic functional $J(\gamma, \sigma)$.

For, by Lemma 9:1 there exists a value of σ such that $k(\sigma) = 0$. Hence, as was noted in the proof of Lemma 9:2 there exist $k(0)$ characteristic values which are less than $\sigma = 0$, that is, the index of $J(\gamma)$ is equal to the number of negative characteristic values of $J(\gamma, \sigma)$.

10. A pseudo-periodic problem. We shall suppose that there exists a constant τ and a set of n^2 constants A_{ij} with $\det A = |A_{ij}| \neq 0$ such that the equations

$$(10:1) \quad 2\omega(x, \gamma, \gamma') = 2\omega(x + \tau, u, u'),$$

$$(10:2) \quad \Phi_\nu(x, \gamma, \gamma') = \Phi_\nu(x + \tau, u, u')$$

are identities in x, γ_1, γ'_1 , where

$$u_1 = A_{1j} \gamma_j, \quad u_1' = A_{1j} \gamma_j'.$$

As a result of these identities the functions L_1 in the differential equations,

$$L_1(x, \gamma, \gamma', \gamma'', \lambda) = \Omega \gamma_1 - \frac{d}{dx} \Omega \gamma_1' = 0,$$

have the property that

$$(10:3) \quad L_1(x, \gamma, \gamma', \gamma'', \lambda) = A_{j1} L_1(x + \tau, u, u', u'', \lambda)$$

identically in $x, \gamma_1, \gamma_1', \gamma_1''$, where $u_1'' = A_{1j} \gamma_j''$.

An extremal arc satisfying the conditions

$$(10:4) \quad \gamma_1(x + \tau) = A_{1j} \gamma_j(x), \quad \gamma_1(x) = A_{j1} \gamma_1(x + \tau)$$

will be called a pseudo-periodic arc of period τ . As a consequence of the preceding identities any extremal arc satisfying (10:4) for one value of x satisfies these relations for all values of x .

The identities (10:1), (10:2), and (10:3) are still valid if we replace τ by $\alpha\tau$ and A_{1j} by A_{1j}^α , where A_{1j}^α are the elements in the matrix A^α . We shall call the problem $P_{\alpha\tau}$ determined by the functional

$$J(\gamma) = \int_0^{\alpha\tau} 2\omega(x, \gamma, \gamma') dx$$

and a maximal class $\mathcal{C}_{\alpha\tau}$ of admissible arcs satisfying the conditions

$$\begin{aligned} \Phi_j(x, \gamma, \gamma') &= 0, \\ \gamma_1(\alpha\tau) &= A_{1j}^\alpha \gamma_j(0) \end{aligned}$$

a pseudo-periodic problem. If the matrix with elements A_{1j}^α is the identity matrix then $P_{\alpha\tau}$ is the usual periodic problem. The transversality conditions for $P_{\alpha\tau}$ are given by

$$J_1(0) = A_{j1}^\alpha J_1(\alpha\tau).$$

It is seen that these conditions follow directly from (10:1).

THEOREM 10:1. The number of conjugate intervals of $(h-1)\tau$ on $(h-1)\tau < x < h\tau$ is the same for all integral values of h .

This follows as a consequence of the identities (10:3) and the fact that the conjugate intervals are unchanged by the transformation $u_1 = A_{1j}^\alpha z_j$.

THEOREM 10:2. The number of conjugate intervals on $0 < x < p\tau$ is equal to p times the number of conjugate intervals on $0 < x < \tau$ plus the number of broken extremal arcs in a maximal set of arcs vanishing at 0 and $p\tau$, having corners at the points $h\tau$ ($h = 1, \dots, p-1$) every linear combination of which has $J \leq 0$ and such that no proper linear combination is an extremal arc.

To prove this we consider the problem P^* determined by a functional of the form (10:1) defined on the interval $0 \leq x \leq p\tau$ and a set $\mathcal{E}^*$ of admissible arcs vanishing at $h\tau$ ($h = 0, \dots, p$) and satisfying $\Phi_\nu = 0$. As a consequence of Theorem 7:1 the index of P^* is equal to p times the number of conjugate intervals on $0 < x < \tau$. Since P^* is a sub-problem of a problem determined by a set of admissible arcs satisfying $\Phi_\nu = 0$ and vanishing at 0 and $p\tau$, the conclusion of the theorem follows as a consequence of Theorem 7:2.

THEOREM 10:3. The index k of $P_{\alpha\tau}$ is given by the formula

$$k = r_0 + r_1 + \dots + r_{\alpha-1} + q,$$

where r_h ($h = 0, \dots, \alpha-1$) is equal to the number of conjugate intervals of $x = h\tau$ on $h\tau < x < (h+1)\tau$ and q is equal to the number of broken extremal arcs with corners at $x = (h+1)\tau$ ($h = 0, \dots, \alpha-2$), in a maximal set of arcs in $\mathcal{E}_{\alpha\tau}$ every linear combination η of which has $J(\eta) \leq 0$, no proper linear combination

of which is a pseudo-periodic arc for $P_{\alpha\tau}$.

LEMMA 10:1. If $\gamma_{1k}, \int_{1k}$, is a conjugate system of extremals and

$$u_{1k}(x + \alpha\tau) = A_{1j}^{\alpha} \gamma_{jk}(x), \quad \int_{1k}(x) = A_{ji}^{\alpha} v_{jk}(x + \alpha\tau),$$

then the functions u_{1k}, v_{1k} define a conjugate system. Moreover, the number of focal intervals of u_{1k} on the interval $x_1 + \alpha\tau < x \leq x_2 + \alpha\tau$ is equal to the number of focal intervals of γ_{1k} on $x_1 < x \leq x_2$.

The proof of the first part of the lemma as given by Birkhoff and Hestenes (I, 241) is applicable in this case and is as follows. By virtue of (10:3), with τ replaced by $\alpha\tau$, the arcs u_{1k} are extremal arcs. Also

$$\gamma_{ij}(x) \int_{1k}(x) = u_{ij}(x + \alpha\tau) v_{1k}(x + \alpha\tau),$$

and hence

$$0 = \gamma_{ij} \int_{1k} - \gamma_{1k} \int_{ij} = u_{ij} v_{1k} - u_{1k} v_{ij}.$$

It follows that the extremals u_{1k}, v_{1k} form a conjugate system. The last statement of the lemma is clearly true.

Let $\gamma_{1k}(x)$ be the extremal arcs of a conjugate system $\gamma_{1k}, \int_{1k}$ with $t_h(x_1)$ secondary focal points on the interval $x_1 < x \leq x_1 + h\tau$. We then have

THEOREM 10:4. The limit

$$\mu = \lim_{h \rightarrow \infty} \frac{t_h(x_1)}{h}$$

exists, is independent of the conjugate system $\gamma_{1k}, \int_{1k}$ and the initial point x_1 . Moreover, the inequalities

$$(10:5) \quad |h\mu - t_h(x_1)| \leq n \quad (h = 1, 2, \dots)$$

are true for all values of x_1 . If $\eta_{1k}(x_1) = 0$, then

$$(10:6) \quad t_h(x_1) \leq h\mu \leq t_h(x_1) + n \quad (h = 1, 2, \dots).$$

To prove this let $\alpha = hk$ and

$$u_{1k}(x + (\alpha - 1)\tau) = A_{1j}^{\alpha-1} \eta_{jk}(x).$$

By Lemma 10:1 the number of focal intervals of the system u_{1k} on $x_1 + (\alpha - h)\tau < x \leq x_1 + \alpha\tau$ is equal to $t_h(x_1)$. By Theorem 8:7 the number of focal intervals of η_{1k} on this interval differs from $t_h(x_1)$ by at most n . Since this statement holds for all values of the integer k in $\alpha = hk$ we have the following relations:

$$(10:7) \quad \begin{aligned} |t_\alpha(x_1) - t_{(k-1)h}(x_1) - t_h(x_1)| &\leq n, \\ |t_{(k-1)h}(x_1) - t_{(k-2)h}(x_1) - t_h(x_1)| &\leq n, \\ &\dots \dots \dots \\ |t_{2h}(x_1) - t_h(x_1) - t_h(x_1)| &\leq n. \end{aligned}$$

On adding these inequalities we have

$$|t_\alpha - kt_h(x_1)| \leq (k-1)n < kn.$$

Similarly

$$|t_\alpha - ht_k(x_1)| < hn.$$

On dividing by $\alpha = hk$ we have

$$(10:8) \quad \begin{aligned} 0 &\leq \left| \frac{t_\alpha(x_1)}{\alpha} - \frac{t_h(x_1)}{h} \right| \leq \frac{n}{h}, \\ 0 &\leq \left| \frac{t_\alpha(x_1)}{\alpha} - \frac{t_k(x_1)}{k} \right| \leq \frac{n}{k}. \end{aligned}$$

Hence

$$\left| \frac{t_h(x_1)}{h} - \frac{t_k(x_1)}{k} \right| \leq \frac{n}{h} + \frac{n}{k}.$$

These relations hold for all values of x_1 . By taking h and k sufficiently large the sum $\frac{n}{h} + \frac{n}{k}$ can be made arbitrarily small. Hence $\mu = \lim_{h \rightarrow \infty} \frac{t_h(x_1)}{h}$ exists and is the same for all values of x_1 . Moreover if μ_1, μ_2 denote the limits of μ and $t_{1h}(x_1), t_{2h}(x_1)$ denote the numbers $t_h(x_1)$ for two different conjugate systems then we have

$$|\mu_1 - \mu_2| = \lim_{h \rightarrow \infty} \frac{|t_{1h}(x_1) - t_{2h}(x_1)|}{h} \leq \lim_{h \rightarrow \infty} \frac{n}{h} = 0.$$

It follows that the constant μ is the same for all conjugate systems. This constant will be called the frequency number. If we let k become infinite in the first of the relations (10:8) and note that $\lim_{k \rightarrow \infty} \frac{t_{\alpha}(x_1)}{\alpha} = \mu$ we get the inequality (10:5). If $\gamma_{1k}(x_1) = 0$ then the number equivalent to q_2 in Lemma 6:5 is equal to zero. It follows that the expressions (10:7) and hence (10:8) hold when the absolute value signs are removed. By letting k become infinite we obtain the relations (10:6).

COROLLARY. If there exists a conjugate system γ_{1k}, γ_{2k} with γ_{1k}, γ_{2k} periodic, then the frequency number μ is rational.

For, if h is suitably chosen then

$$\gamma_{1k}(x + (\alpha - 1)\tau) = A_{1j}^{\alpha-1} \gamma_{jk}(x),$$

where $\alpha = hk$ and $k = 1, 2, \dots$. It follows that the left members of the inequalities (10:7) are equal to zero and hence

$$h\mu - t_h(x_1) = 0.$$

The conclusion of the corollary is immediate.

Finally let us consider the periodic problem with period τ where the matrix A is the identity matrix and let P be the problem defined on the interval $0 \leq x \leq \tau$ with end conditions

$\gamma_1(0) = \gamma_1(\tau) = 0$. Let P^* be the sub-problem of P with arcs

vanishing at $x = \tau$. This is a problem of the type discussed in Section 7 and the index of P with respect to P^* is the number of broken extremal arcs in a maximal set with corners at τ , vanishing at $x = 0$ and $x = r\tau$, every linear combination of which having $J(\gamma) \leq 0$, but no proper linear combination being a periodic extremal arc. This number is the same as the r th index of repletion as defined by Morse and Pitcher (X). They consider only the case when the points $x = \tau$ and $x = r\tau$ are not conjugate to each other nor to $x = 0$.

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VITA

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